Chapter 2: Introduction to Stiffness Method

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Aims of this lecture

- A review on the concept of stiffness matrix
- Derivation of stiffness matrix for spring element by
  - Direct equilibrium method
  - Minimum potential energy method

Review to Concept of Stiffness Matrix

Step1: Definition of Element Type & Discretization

Step2: Select a Displacement Function

Step3: Define Stain/Displacement & Stress/Strain Relationships

Step4: Derive the Element Stiffness Matrix & Equations

Step5: Assemble Element Equations & Impose Boundary Conditions

Step6: Solve for Unknown Degree-of-Freedom

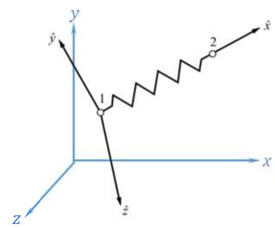
Step7: Solve for Element Strains & Stresses

Local nodal force

Local stiffness matrix: $\{\hat{f}\} = [\hat{k}]\{\hat{d}\}$

Local nodal displacement

Local coordinate system of element: $(\hat{x}, \hat{y}, \hat{z})$



Global coordinate system of element: (x, y, z)

Global nodal force

Global stiffness matrix: $\{F\} = [K]\{d\}$

Global nodal displacement

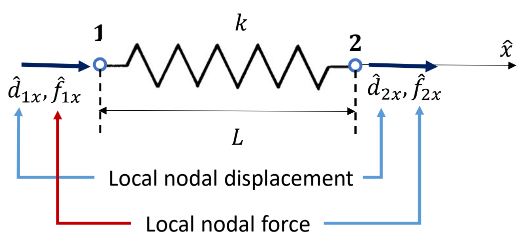
Stiffness Matrix for a Spring Element

- The simplest element:

Properties of spring:
- An initial length (L)
- Obeys Hooke's law (Stiffness constant of k)
- Resists force only along its axis

There are two reference points at both ends of spring element called **nodes**.

Since the forces exert along the axis, we need to span a local axis along spring element.



$$\{\hat{f}\} = [\hat{k}]\{\hat{d}\}$$

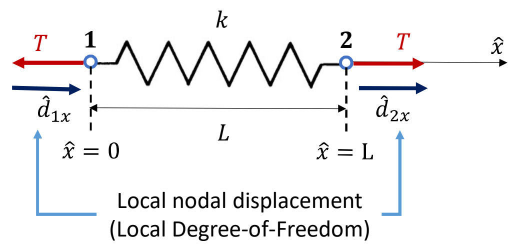
$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = [\hat{k}] \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \boxed{\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

Local stiffness matrix for spring element

Derivation of Stiffness Matrix

- Step 1: Select the Element Type
  - There are 2 nodes for the element
  - Nodes should be labeled
  - The material property of the element is k
  - The length of element is L

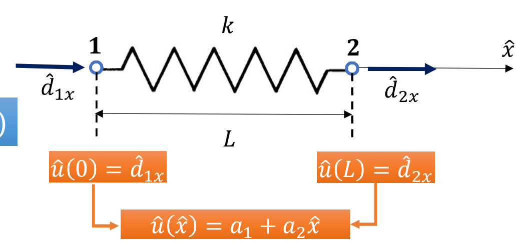


Derivation of Stiffness Matrix

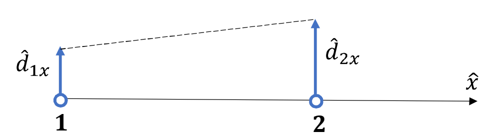
- Step 2: Select a Displacement Function
  - The mathematical function to represent the deformed shape of the spring element under loading.
  - The most common function used are polynomial.
  - Local DOFs are along $\hat{x}$

The displacement function is chosen to be $\hat{u}(\hat{x})$

A unique linear function can be used to describe displacement in element under loading according to nodal displacements.



The total number of coefficients a is equal to the total number of degrees of freedom associated with the element.



Derivation of Stiffness Matrix

$$\hat{u}(\hat{x}) = a_1 + a_2\hat{x} \quad \Rightarrow \quad \hat{u}(\hat{x}) = \begin{bmatrix} 1 & \hat{x} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}$$

Boundary Conditions:

$$\hat{u}(0) = \hat{d}_{1x} \rightarrow \hat{u}(0) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = a_1 = \hat{d}_{1x}$$

$$\hat{u}(L) = \hat{d}_{2x} \rightarrow \hat{u}(L) = \begin{bmatrix} 1 & L \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = a_1 + a_2 L = \hat{d}_{2x}$$

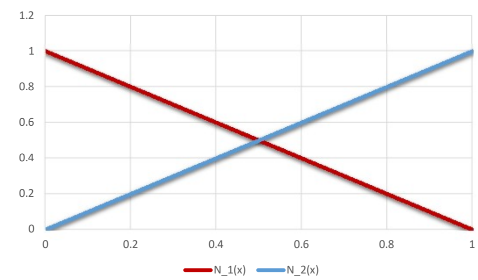
$$a_1 = \hat{d}_{1x}$$

$$a_2 = \frac{\hat{d}_{2x} - \hat{d}_{1x}}{L}$$

$$\hat{u}(\hat{x}) = \hat{d}_{1x} + \frac{\hat{d}_{2x} - \hat{d}_{1x}}{L}\hat{x} = \underbrace{\boxed{\left(1 - \frac{\hat{x}}{L}\right)}}_{N_1(\hat{x})}\hat{d}_{1x} + \underbrace{\boxed{\frac{\hat{x}}{L}}}_{N_2(\hat{x})}\hat{d}_{2x}$$

$$\hat{u}(\hat{x}) = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

Shape Functions
(Interpolation Functions)



Properties of Shape Functions

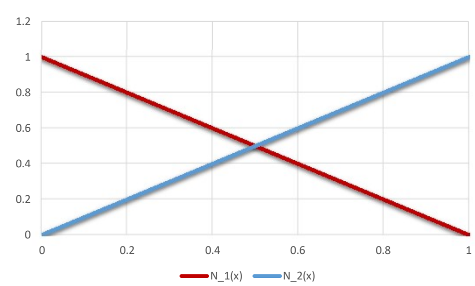
$$N_1(0) = 1 \qquad N_1(L) = 0$$

$$N_2(0) = 0 \qquad N_2(L) = 1$$

Shape function corresponding to node i, is equal to 1 at node i and equal to 0 at any other node.

$$N_1 + N_2 = \left(1 - \frac{\hat{x}}{L}\right) + \frac{\hat{x}}{L} = 1$$

The interpolation function may be different from the actual function except at the endpoints or nodes

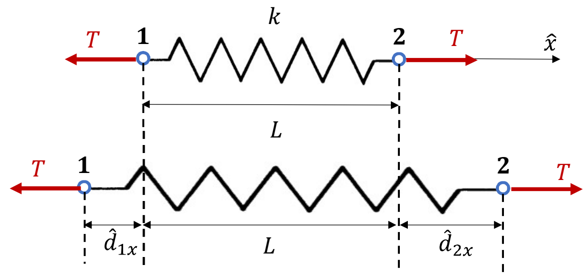


Derivation of Stiffness Matrix

- Step 3 :Define the Strain/Displacement & Stress/Strain Relationships

The deformation of the spring is:

$$\delta = \hat{u}(L) - \hat{u}(0) = \hat{d}_{2x} - \hat{d}_{1x}$$



The strain of the spring is:

$$\varepsilon = \frac{\delta}{L} = \frac{\hat{u}(L) - \hat{u}(0)}{L} = \frac{\hat{d}_{2x} - \hat{d}_{1x}}{L}$$

The relation between force and displacement in the spring is:

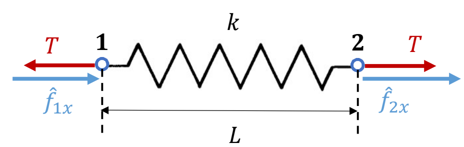
$$T = k\delta = k(\hat{d}_{2x} - \hat{d}_{1x})$$

Derivation of Stiffness Matrix

- Step 4: Derive the Element Stiffness Matrix and Equations

Direct Equilibrium
Minimum Potential Energy
Weighted Residual Method

$$T = -\hat{f}_{1x}$$
$$T = +\hat{f}_{2x}$$



$$T = k\delta = k(\hat{d}_{2x} - \hat{d}_{1x})$$

$$-\hat{f}_{1x} = k(\hat{d}_{2x} - \hat{d}_{1x})$$
$$+\hat{f}_{2x} = k(\hat{d}_{2x} - \hat{d}_{1x})$$

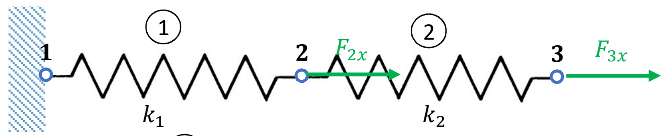
$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \boxed{\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

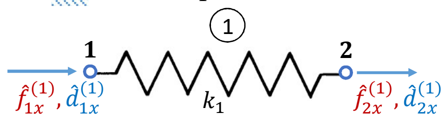
Local stiffness matrix for spring element

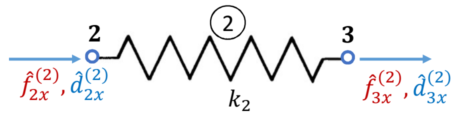
The matrix is Square, Symmetry and Singular

Derivation of Stiffness Matrix

- Step 5: Assemble the Element Equations to Obtain the Global Equations and Introduce Boundary Conditions







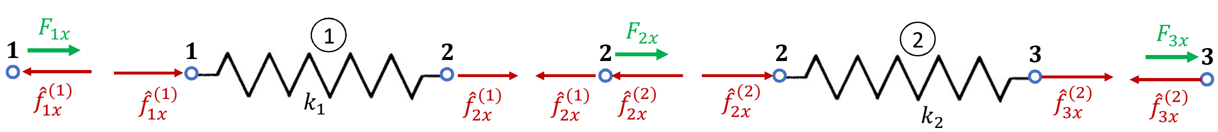
$$\begin{Bmatrix} \hat{f}_{1x}^{(1)} \\ \hat{f}_{2x}^{(1)} \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x}^{(1)} \\ \boxed{\hat{d}_{2x}^{(1)}} \end{Bmatrix} \begin{matrix} d_{1x} \\ d_{2x} \end{matrix}$$

Compatibility

$$\begin{Bmatrix} \hat{f}_{2x}^{(2)} \\ \hat{f}_{3x}^{(2)} \end{Bmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} \boxed{\hat{d}_{2x}^{(2)}} \\ \hat{d}_{3x}^{(2)} \end{Bmatrix} \begin{matrix} d_{2x} \\ d_{3x} \end{matrix}$$

Compatibility

Derivation of Stiffness Matrix-Step 5



$$\begin{Bmatrix} \hat{f}_{1x}^{(1)} \\ \hat{f}_{2x}^{(1)} \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix} \qquad \begin{Bmatrix} \hat{f}_{2x}^{(2)} \\ \hat{f}_{3x}^{(2)} \end{Bmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} d_{2x} \\ d_{3x} \end{Bmatrix}$$

Equilibrium @ node#1 $\hat{f}_{1x}^{(1)} - \mathrm{F}_{1x} = 0$

Equilibrium @ node#2 $\hat{f}_{2x}^{(1)} + \hat{f}_{2x}^{(2)} - \mathrm{F}_{2x} = 0$

Equilibrium @ node#3 $\hat{f}_{3x}^{(2)} - \mathrm{F}_{3x} = 0$

$\Rightarrow$

$$\begin{aligned} F_{1x} &= k_1 d_{1x} - k_1 d_{2x} \\ F_{2x} &= -k_1 d_{1x} + k_1 d_{2x} + k_2 d_{2x} - k_2 d_{3x} \\ F_{3x} &= -k_2 d_{2x} + k_2 d_{3x} \end{aligned}$$

Derivation of Stiffness Matrix-Step 5

Singular Matrix

Global nodal force vector
$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{Bmatrix}$$
Global nodal displacement vector

$[K]$: Global Stiffness Matrix
(Total Stiffness Matrix)

Assembling Stiffness Matrices by Superposition Method

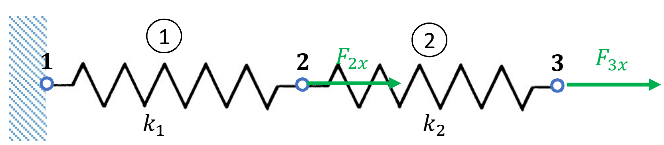
Stiffness matrix for element#1

Stiffness matrix for element#2

$$\begin{array}{cccc} d_{1x} & d_{2x} & d_{3x} & \\ \hline \end{array}$$
$$\begin{array}{ccc} d_{1x} & d_{2x} & d_{3x} \end{array}$$
$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{matrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{matrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{matrix} \Rightarrow \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}$$

Derivation of Stiffness Matrix-Step 5

- Boundary Conditions
  - Homogeneous
  - Non-homogeneous



Homogeneous Boundary Conditions: $d_{1x} = 0$

$$\boxed{F_{1x}} = k_1(0) - k_1\boxed{d_{2x}}$$
$$F_{2x} = -k_1(0) + (k_1+k_2)\boxed{d_{2x}} - k_2\boxed{d_{3x}}$$
$$F_{3x} = -k_2\boxed{d_{2x}} + k_2\boxed{d_{3x}}$$

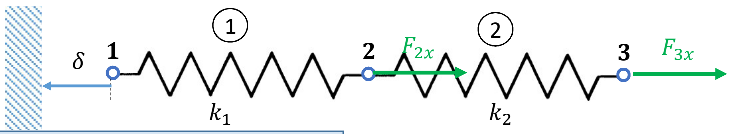
$$\Rightarrow \begin{Bmatrix} F_{2x} \\ F_{3x} \end{Bmatrix} = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \boxed{\begin{Bmatrix} d_{2x} \\ d_{3x} \end{Bmatrix}}$$

Not singular

Unknowns

Derivation of Stiffness Matrix-Step 5

Non-homogeneous Boundary Conditions: $d_{1x} = \delta$



$$\boxed{F_{1x}} = k_1(\delta) - k_1\boxed{d_{2x}}$$
$$F_{2x} = -k_1(\delta) + (k_1 + k_2)\boxed{d_{2x}} - k_2\boxed{d_{3x}}$$
$$F_{3x} = -k_2\boxed{d_{2x}} + k_2\boxed{d_{3x}}$$

$$\Rightarrow \begin{Bmatrix} F_{2x} + k_1\delta \\ F_{3x} + (0)\delta \end{Bmatrix} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \boxed{\begin{Bmatrix} d_{2x} \\ d_{3x} \end{Bmatrix}}$$

Not singular

Unknowns

Derivation of Stiffness Matrix-Step 5

Remove rows & columns corresponding to known DOFs

Subtract known DOF multiplied by corresponding column from the force vector

Solve for reduced system of equations

Using obtained DOFs, solve for unknown forces

$$-\delta\begin{Bmatrix} -k_1 \\ 0 \end{Bmatrix} + \begin{Bmatrix} F_{2x} \\ F_{3x} \end{Bmatrix} = \begin{bmatrix} k_1 & & \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} d_{2x} \\ d_{3x} \end{Bmatrix}$$

Derivation of Stiffness Matrix by Minimum Potential Energy

- Step 4: Derive the Element Stiffness Matrix and Equations

Direct Equilibrium

Minimum Potential Energy

Weighted Residual Method

more adaptable to the determination of element equations for complicated elements

Only applicable for elastic materials

Total Potential Energy in a system is a function of displacements

$$\pi_p \equiv \pi_p(d_1, d_2, \ldots, d_n)$$

When π_p is minimized with respect to these displacements, equilibrium equations result.

Derivation of Stiffness Matrix by Minimum Potential Energy

- The principle of minimum potential energy:

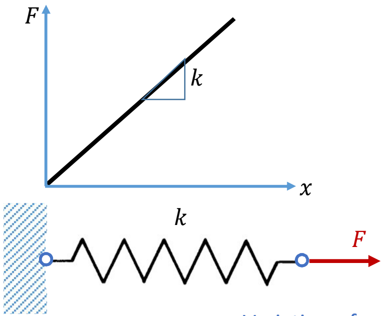
Of all the geometrically possible shapes that a body can assume, the true one, corresponding to the satisfaction of stable equilibrium of the body, is identified by a minimum value of the total potential energy

$$\pi_p = \boxed{U} + \boxed{\Omega}$$

Internal Strain Energy:
The capacity of internal forces (or stresses) to do work through deformations (strains) in the structure

Potential Energy of External Forces:
The capacity of external forces to do work through deformation of the structure.

Derivation of Stiffness Matrix by Minimum Potential Energy



$$dU = Fdx$$
$$F = kx$$
$$dU = kxdx$$

$$U = \frac{1}{2}kx^2$$

$$\Omega = -Fx$$

$$\pi_p = \frac{1}{2}kx^2 - Fx$$

AT stable equilibrium, a minimum for Potential energy is met.

At minimum point of potential energy: $\delta\pi_p = 0$

$$\boxed{\delta\pi_p} = \frac{\partial\pi_p}{\partial d_1}\delta d_1 + \frac{\partial\pi_p}{\partial d_2}\delta d_2 + \cdots + \frac{\partial\pi_p}{\partial d_n}\boxed{\delta d_n} = 0$$

Variation of potential energy

Variation of displacement at n

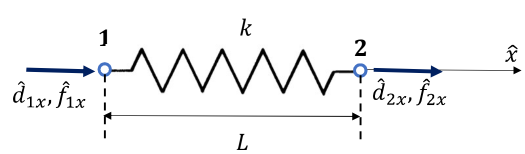
At equilibrium, above statement is correct for every arbitrary variations of displacement

$$\frac{\partial\pi_p}{\partial d_i} = 0, \qquad i = 1, \dots, n$$

Element equations

Derivation of Stiffness Matrix by Minimum Potential Energy

$$\pi_p = \frac{1}{2}kx^2 - Fx$$



$$\pi_p = \frac{1}{2}k(\hat{d}_{2x} - \hat{d}_{1x})^2 - \hat{f}_{1x}\hat{d}_{1x} - \hat{f}_{2x}\hat{d}_{2x}$$

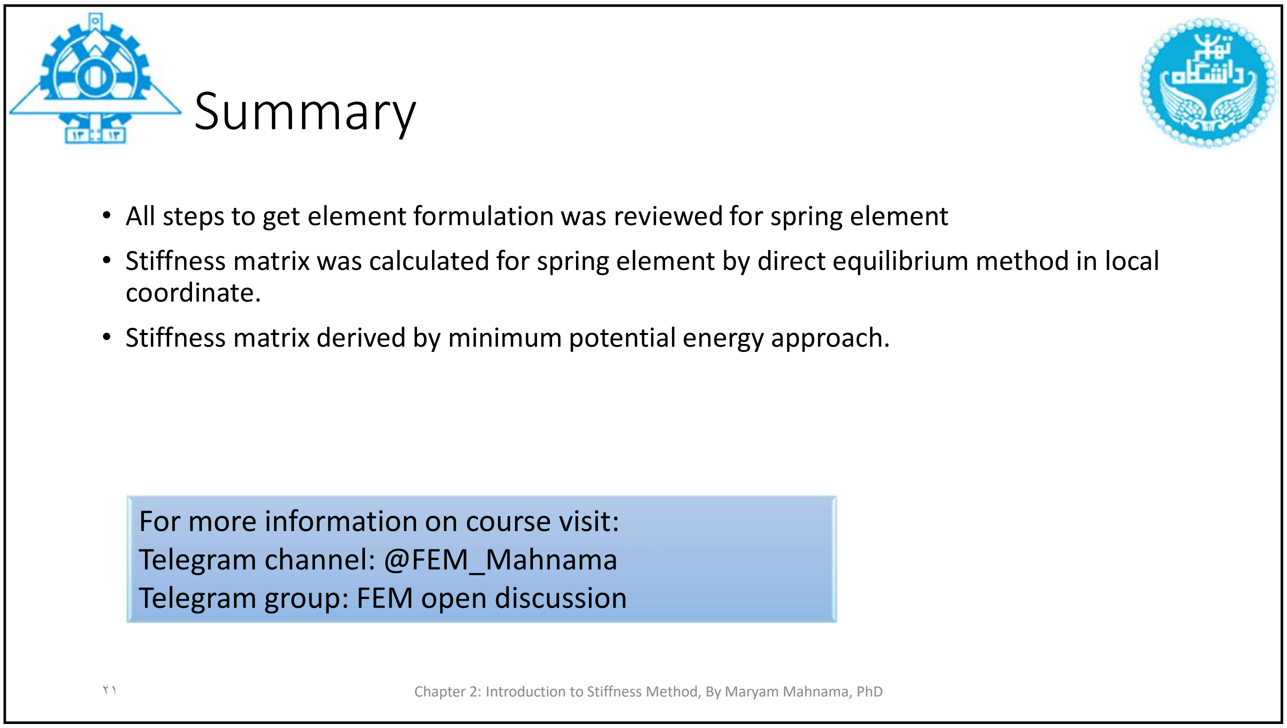
$$\frac{\partial\pi_p}{\partial\hat{d}_{1x}} = -k(\hat{d}_{2x} - \hat{d}_{1x}) - \hat{f}_{1x} = 0$$

$$\frac{\partial\pi_p}{\partial\hat{d}_{2x}} = +k(\hat{d}_{2x} - \hat{d}_{1x}) - \hat{f}_{2x} = 0$$

$$-\hat{f}_{1x} = k(\hat{d}_{2x} - \hat{d}_{1x})$$
$$+\hat{f}_{2x} = k(\hat{d}_{2x} - \hat{d}_{1x})$$

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

Local stiffness matrix for spring element


Summary
• All steps to get element formulation was reviewed for spring element
• Stiffness matrix was calculated for spring element by direct equilibrium method in local coordinate.
• Stiffness matrix derived by minimum potential energy approach.
For more information on course visit:
Telegram channel: @FEM_Mahnama
Telegram group: FEM open discussion
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Chapter 2: Introduction to Stiffness Method, By Maryam Mahnama, PhD