Chapter 3: Bar (Truss) Element

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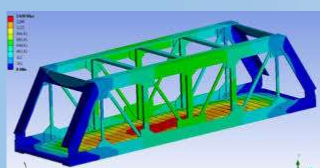


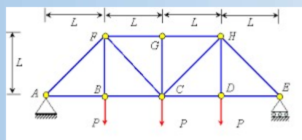


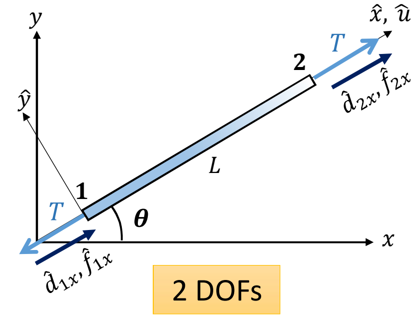
Figure T4.11

Aims of this lecture

- Derivation of Stiffness matrix for Bar element in local coordinates
- Transformation of vectors in 2D
- Global stiffness matrix for bar element
- Stress Computation in Bar element in 2D

Introduction to Bar element

- An arbitrary oriented Bar in global coordinates system: (x, y)
- Introducing a local coordinates system: $(\hat{x}, \hat{y})$
- The Bar element has a constant:
  - Cross section area A
  - Young's modulus E
  - Length L
- From Hooke's law: $\sigma_x = E\epsilon_x = E\dfrac{d\hat{u}}{d\hat{x}}$
- From Equilibrium: $\mathrm{T} = A\sigma_x = constant$



Differential Equation for Bar element

$$\frac{d}{d\hat{x}}\left(AE\frac{d\hat{u}}{d\hat{x}}\right) = 0$$

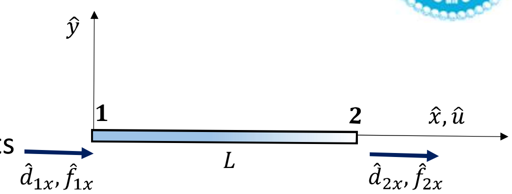
Introduction to Bar element

Stiffness matrix in local coordinates system → Transformation → Stiffness matrix in global coordinates system

Step 1: Define Element Type
Step 2: Select Displacement Function
Step 3: Strain-Displacement and Stress-Strain Relationships
Step 4: Derive the Element Stiffness Matrix and Equations

Assemble Element Equations to Obtain Global or Total Equations

Stiffness matrix for Bar element in local coordinates



Step 2: Select Displacement Function

- An element with 2 DOFs ⇒ 2 nodal displacements as B.C.s
- $\hat{u} = a_0 + a_1\hat{x}$

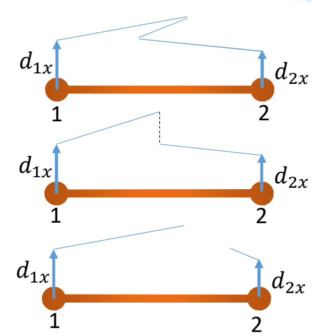
B.C.s: $\hat{u}(0) = \hat{d}_{1x}, \quad \hat{u}(L) = \hat{d}_{2x}$

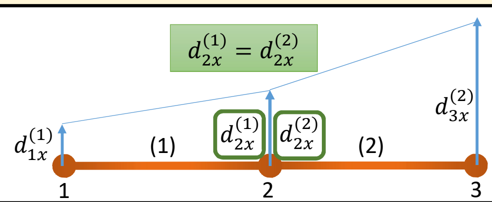
$$\hat{u} = \hat{d}_{1x} + \left(\frac{\hat{d}_{2x} - \hat{d}_{1x}}{L}\right)\hat{x}$$

$$\hat{u} = \underbrace{\boxed{\left(1 - \frac{\hat{x}}{L}\right)}}_{N_1(\hat{x})}\hat{d}_{1x} + \underbrace{\boxed{\frac{\hat{x}}{L}}}_{N_2(\hat{x})}\hat{d}_{2x} \quad \Rightarrow \quad \hat{u} = \underbrace{\boxed{[N_1 \quad N_2]}}_{[N]}\underbrace{\boxed{\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}}}_{\{d\}}$$

Selecting Displacement Functions

- The approximate (displacement) functions
  1. are commonly polynomials.
  2. are continuous within the element (No opening, overlap or jump)
  3. provide interelement continuity for all degrees of freedom at each node





The linear function is then called a **conforming**, or **compatible**, function for the bar element.

Selecting Displacement Functions

- According to inter-element continuity of elements, we can categorize the elements.
- The symbol C^m is used to describe the continuity of a piecewise field such as axial displacement.

C^m

- m: Degree of derivative that is interelement continuous
- C: Stands for continuity

Example: Bar element is a C^0 element, because only the displacement function itself is continuous at the element boundary.

If the field variable and its first derivative are continuous across the common boundary, then the field variable is said to be C^1 continuous.

Selecting Displacement Functions

- The approximate (displacement) functions (continued)

4. allow for rigid-body displacement and for a state of constant strain within the element.

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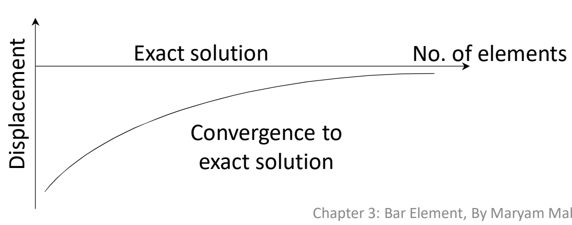
In the displacement function, lower-order terms cannot be omitted in favor of the higher-order term **Completeness**

Completeness of a function is a necessary condition for convergence to the exact answer



Stiffness matrix for Bar element in local coordinates

Step 3: Strain-Displacement and Stress-Strain Relationships

- The strain/displacement relationship

$$\epsilon_x = \frac{d\hat{u}}{d\hat{x}} = \frac{d[N]}{d\hat{x}}\{d\}$$

$$\epsilon_x = \left(\frac{\hat{d}_{2x} - \hat{d}_{1x}}{L}\right)$$

$$= \underbrace{\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}}_{[B]} \underbrace{\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}}_{\{d\}}$$

- stress/strain relationship

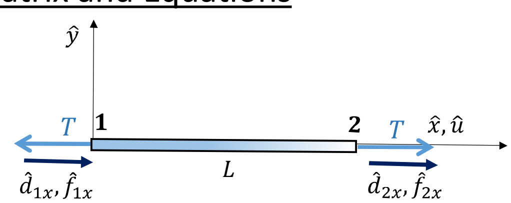
$$\sigma_x = E\epsilon_x$$

$$\sigma_x = E[B]\{d\}$$

Stiffness matrix for Bar element in local coordinates

Step 4: Derive the Element Stiffness Matrix and Equations

- Direct Approach
- Minimum Potential Energy
- Galerkin's Method



$$\mathrm{T} = A\sigma_x = AE[B]\{d\} = AE\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

$$\mathrm{T} = -\hat{f}_{1x} \Rightarrow \hat{f}_{1x} = \frac{AE}{L}(+\hat{d}_{1x} - \hat{d}_{2x})$$

$$\mathrm{T} = \hat{f}_{2x} \Rightarrow \hat{f}_{2x} = \frac{AE}{L}(-\hat{d}_{1x} + \hat{d}_{2x})$$

Bar element stiffness matrix in local coordinates

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \underbrace{\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{[\hat{k}]} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

Stiffness matrix for Bar element in local coordinate

Stiffness matrix in local coordinates system → Transformation → Stiffness matrix in global coordinates system

- Step 1: Define Element Type
- Step 2: Select Displacement Function
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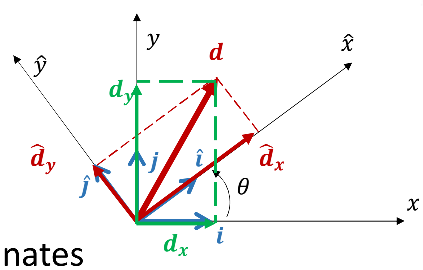
- Step 4: Derive the Element Stiffness Matrix and Equations

Assemble Element Equations to Obtain Global or Total Equations

Transformation

- Assumption:
  - Global coordinates $(x, y) \Rightarrow$ unit vectors: $\boldsymbol{i}, \boldsymbol{j}$
  - Local coordinates $(\hat{x}, \hat{y}) \Rightarrow$ unit vectors: $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}$

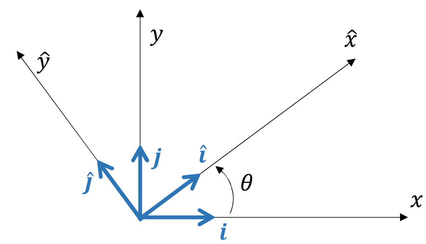


- vector d can be expressed in these coordinates
- $\boldsymbol{d} = d_x\boldsymbol{i} + d_y\mathbf{j}$
- $\boldsymbol{d} = \hat{d}_x\hat{\boldsymbol{\imath}} + \hat{d}_y\hat{\boldsymbol{\jmath}}$

$$\boldsymbol{d} = d_x\boldsymbol{i} + d_y\mathbf{j} = \hat{d}_x\hat{\boldsymbol{\imath}} + \hat{d}_y\hat{\boldsymbol{\jmath}}$$

What is the relationship between local and global coordinates?

Transformation



$$\hat{\boldsymbol{\imath}}.(d_x\boldsymbol{i} + d_y\mathbf{j}) = \hat{\boldsymbol{\imath}}.(\hat{d}_x\hat{\boldsymbol{\imath}} + \hat{d}_y\hat{\boldsymbol{\jmath}})$$

$$\hat{d}_x = (d_x\hat{\mathbf{\imath}}.\boldsymbol{i} + d_y\hat{\mathbf{\imath}}.\mathbf{j})$$

$$\hat{\boldsymbol{\imath}}.\boldsymbol{i} = 1 \times 1 \times \cos\theta = \cos\theta$$

$$\hat{\boldsymbol{\imath}}.\mathbf{j} = 1 \times 1 \times \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\hat{d}_x = (d_x \cos\theta + d_y \sin\theta)$$

$$\hat{\boldsymbol{\jmath}}.(d_x\boldsymbol{i} + d_y\mathbf{j}) = \boldsymbol{j}.(\hat{d}_x\hat{\boldsymbol{\imath}} + \hat{d}_y\hat{\boldsymbol{\jmath}})$$

$$\hat{d}_y = (d_x\hat{\boldsymbol{\jmath}}.\boldsymbol{i} + d_y\hat{\boldsymbol{\jmath}}.\mathbf{j})$$

$$\hat{\boldsymbol{\jmath}}.\boldsymbol{i} = 1 \times 1 \times \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$$

$$\hat{\boldsymbol{\jmath}}.\hat{\boldsymbol{\jmath}} = 1 \times 1 \times \cos\theta = \cos\theta$$

$$\hat{d}_y = (-d_x \sin\theta + d_y \cos\theta)$$

$$\begin{Bmatrix} \hat{d}_x \\ \hat{d}_y \end{Bmatrix} = \begin{bmatrix} C & S \\ -S & C \end{bmatrix} \begin{Bmatrix} d_x \\ d_y \end{Bmatrix}$$

Transformation (rotation) Matrix

$$C \equiv \cos\theta, S \equiv \sin\theta$$

Stiffness matrix for Bar element in local coordinate

Stiffness matrix in local coordinates system → Transformation → Stiffness matrix in global coordinates system

- Step 1: Define Element Type
- Step 2: Select Displacement Function
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- Step 4: Derive the Element Stiffness Matrix and Equations

Assemble Element Equations to Obtain Global or Total Equations

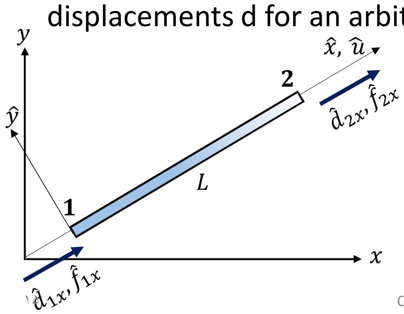
Stiffness matrix for Bar element in global coordinate

- We had before:

Bar element stiffness matrix in local coordinates

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \underbrace{\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{[\hat{k}]} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

- We now want to relate the global element nodal forces f to the global nodal displacements d for an arbitrarily oriented bar element:



Bar element stiffness matrix in global coordinates

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = \boxed{\boldsymbol{k}} \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{Bmatrix}$$

Stiffness matrix for Bar element in global coordinate

- Local coordinates can be expressed as:

$$\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix} = \begin{Bmatrix} d_{1x}\cos\theta + d_{1y}\sin\theta \\ d_{2x}\cos\theta + d_{2y}\sin\theta \end{Bmatrix} = \boxed{\begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix}} \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{Bmatrix} \Rightarrow \boxed{\hat{\boldsymbol{d}} = \boldsymbol{T}^*\boldsymbol{d}}$$

T*: Transformation matrix

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix} \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} \Rightarrow \boxed{\hat{\boldsymbol{f}} = \boldsymbol{T}^*\boldsymbol{f}}$$

$$\hat{\boldsymbol{f}} = \hat{\boldsymbol{k}}\,\hat{\boldsymbol{d}}$$

$$\boxed{\boldsymbol{T}^*\boldsymbol{f}} = \hat{\boldsymbol{k}}\,\boxed{\boldsymbol{T}^*\boldsymbol{d}}$$

$\boldsymbol{T}^*$ is not a square matrix and cannot be inversed

Stiffness matrix for Bar element in global coordinate

$$\begin{Bmatrix}\hat{d}_{1x}\\ \hat{d}_{1y}\\ \hat{d}_{2x}\\ \hat{d}_{2y}\end{Bmatrix}=\begin{Bmatrix}d_{1x}\cos\theta+d_{1y}\sin\theta\\ -d_{1x}\sin\theta+d_{1y}\cos\theta\\ d_{2x}\cos\theta+d_{2y}\sin\theta\\ -d_{2x}\sin\theta+d_{2y}\cos\theta\end{Bmatrix}=\boxed{\begin{bmatrix}C&S&0&0\\-S&C&0&0\\0&0&C&S\\0&0&-S&C\end{bmatrix}}\begin{Bmatrix}d_{1x}\\ d_{1y}\\ d_{2x}\\ d_{2y}\end{Bmatrix}$$

T: Transformation matrix

$\hat{d}=Td$

$$\begin{Bmatrix}\hat{f}_{1x}\\ \hat{f}_{1y}\\ \hat{f}_{2x}\\ \hat{f}_{2y}\end{Bmatrix}=\begin{bmatrix}C&S&0&0\\-S&C&0&0\\0&0&C&S\\0&0&-S&C\end{bmatrix}\begin{Bmatrix}f_{1x}\\ f_{1y}\\ f_{2x}\\ f_{2y}\end{Bmatrix}$$

$\hat{f}=Tf$

$$\hat{f}=\hat{k}\,\hat{d}$$
$$\boxed{Tf}=\hat{k}\boxed{Td}$$

Stiffness matrix for Bar element in global coordinate

$$Tf=\hat{k}\,Td$$
$$T^{-1}Tf=T^{-1}\hat{k}Td$$

Bar element stiffness matrix in global coordinates

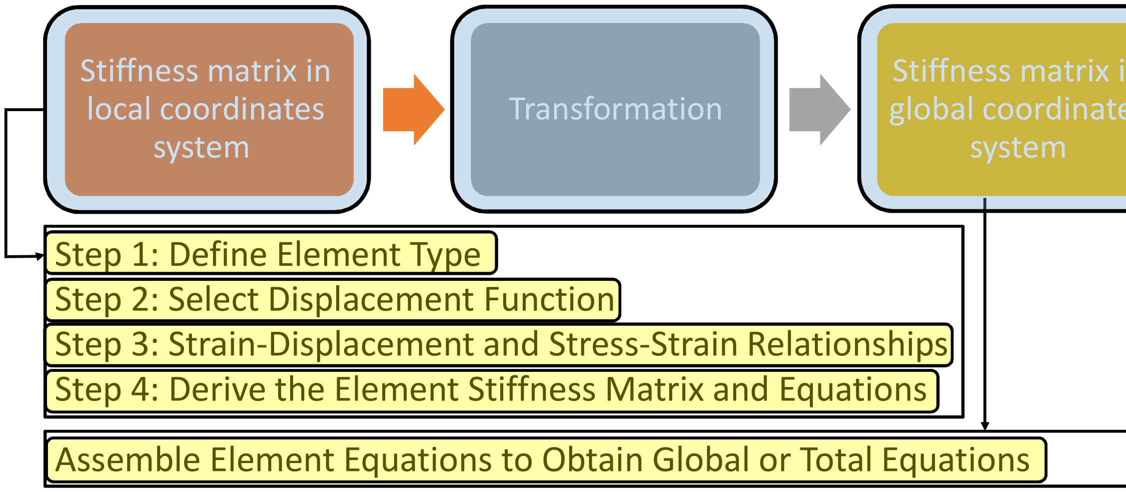
$$f=\underbrace{T^{-1}\hat{k}T}_{[k]}d$$

$$k=T^{T}\hat{k}T$$

It can be shown that [T] is an orthogonal matrix, while dot product of its rows (columns) are zero.
It can be shown that in an orthogonal matrix such as [T], transpose of matrix is equal to its inverse: $[T]^T=[T]^{-1}$

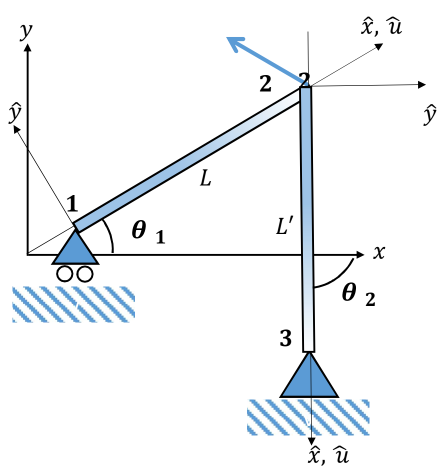
$$[k]=\frac{AE}{L}\begin{bmatrix}C^2&CS&-C^2&-CS\\CS&S^2&-CS&-S^2\\-C^2&-CS&C^2&CS\\-CS&-S^2&CS&S^2\end{bmatrix}$$

Stiffness matrix for Bar element in local coordinate



Assemble Element Equations to Obtain Global or Total Equations

- Element stiffness matrices and force vectors should be transformed into global coordinates.
- Then they can be assembled.



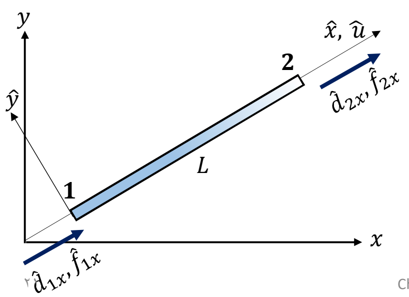
Computation of stress in Bar element

- Local forces result in stress in element

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

- The usual definition of stress in a bar is

$$\sigma_x = \frac{\hat{f}_{2x}}{\mathrm{A}} = \frac{\mathrm{E}}{\mathrm{L}}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$



$$\sigma_x = \frac{\mathrm{E}}{\mathrm{L}}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix}\begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{Bmatrix}$$

$$\sigma_x = \frac{\mathrm{E}}{\mathrm{L}}\begin{bmatrix} -C & -S & C & S \end{bmatrix}\{d\}$$

Summary

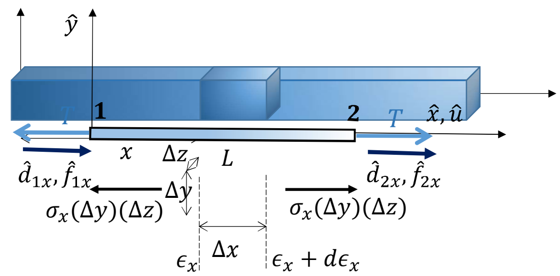
- Bar Element Stiffness Matrix in Local Coordinate
- Concepts of Compatibility, Completeness and Convergence
- Transformation Matrix
- Bar Element Stiffness Matrix in Global Coordinate
- Stress Calculation for Bar Element

Further Readings:
Sections 3-6, 3-7, 3-8 and 3-9 from "A first course in finite element" by Logan

Minimum Potential Energy to Derive Bar Element Equations in Local Coordinates

<u>Step 4: Derive the Element Stiffness Matrix and Equations</u>

- Direct Approach
- **Minimum Potential Energy**
- Galerkin's Method



$$\boxed{\pi_p = U + \Omega}$$

U: internal strain energy

The work done by the internal forces during deformation

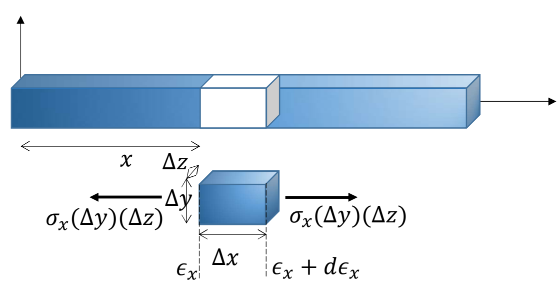
At any infinitesimal part on bar → The internal force in bar element is: $\sigma_x(\Delta y)(\Delta z)$

The displacement at each point x is: $(\epsilon_x + d\epsilon_x)(\Delta x) - \epsilon_x(\Delta x)$

$$dU = \sigma_x(\Delta y)(\Delta z)d\epsilon_x(\Delta x)$$

Minimum Potential Energy to Derive Bar Element Equations in Local Coordinates

$$U = \int dU = \iiint_V \int \sigma_x d\epsilon_x \, dV$$



- $U = \int dU = \iiint_V \int \sigma_x d\epsilon_x \, dV$
- Hooke's law: (for linear elastic material)

$\sigma_x = E\epsilon_x \rightarrow d\sigma_x = E d\epsilon_x$

- $U = \int dU = \frac{1}{2} \iiint_V \sigma_x \epsilon_x dV$

Minimum Potential Energy to Derive Bar Element Equations in Local Coordinates

Step 4: Derive the Element Stiffness Matrix and Equations

- Direct Approach
- Minimum Potential Energy
- Galerkin's Method

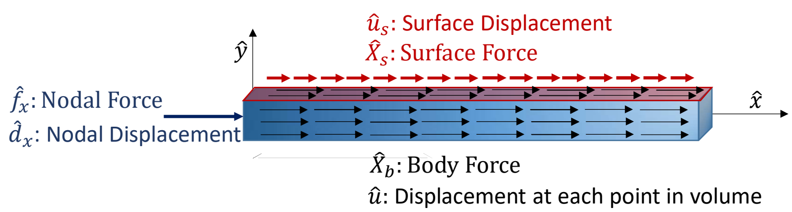
$$\boxed{\pi_p = U + \Omega}$$

Internal Strain Energy

Work of External Forces

Minimum Potential Energy to Derive Bar Element Equations in Local Coordinates

External Forces:
Nodal Force
Surface (Traction) Force
Body Force



$$\Omega = -\sum_{i=1}^{M} \hat{f}_{ix}\hat{d}_{ix} - \iint_{S_1} \hat{X}_s \hat{u}_s \, dS - \iiint_{V} \hat{X}_b \hat{u} \, dV$$

Number of DOFs in the element

Part of the surfa ce of element on which $\hat{X}_s$ is exerted

Volume of the body

Minimum Potential Energy to Derive Bar Element Equations in Local Coordinates

- The finite element process seeks:

A minimum in the potential energy within the constraint of an assumed displacement pattern within each element.

An approximate finite element solution found by using the stiffness method will always provide an approximate value of potential energy greater than or equal to the correct one.

This method also results in a structure behavior that is predicted to be physically stiffer than, or at best to have the same stiffness as, the actual one.

Why?

In FEM we assume a displacement field for the element. The assumed field effectively constrains the structure from deforming in its natural manner. $\Longrightarrow$ This constraint effect stiffens the predicted behavior of the structure.

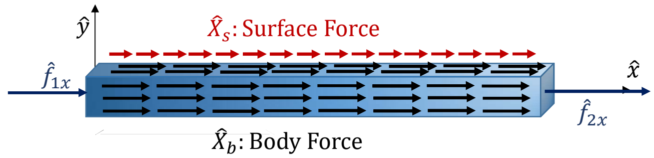
Minimum Potential Energy to Derive Bar Element Equations in Local Coordinates

- Procedure:

1. An expression for the total potential energy.
2. Assume the displacement pattern to be substituted into the expression for total potential energy.
3. Minimizing the total potential energy with respect to these nodal parameters.

These resulting equations represent the element equations

The resulting equations are the approximate (or possibly exact) equilibrium equations.



Step 1

$$\pi_p = \frac{1}{2}\iiint_V \sigma_x \epsilon_x dV - \sum_{i=1}^{M} \hat{f}_{ix}\hat{d}_{ix} - \iint_{S_1} \hat{X}_s \hat{u}_s\, dS - \iiint_V \hat{X}_b \hat{u}\, dV$$

$$\pi_p = \frac{A}{2}\int_0^L \sigma_x \epsilon_x d\hat{x} - \hat{f}_{1x}\hat{d}_{1x} - \hat{f}_{2x}\hat{d}_{2x} - \iint_{S_1} \hat{X}_s \hat{u}_s\, dS - \iiint_V \hat{X}_b \hat{u}\, dV$$

Step 2

$$\hat{u} = [N]\{\hat{d}\} \qquad \hat{u}_s = [N_s]\{\hat{d}\} \qquad \{\epsilon_x\} = [B]\{d\} \qquad [B] = \left[-\frac{1}{L} \quad \frac{1}{L}\right]$$

$$[N] = \left[1 - \frac{\hat{x}}{L} \quad \frac{\hat{x}}{L}\right] \qquad \{\sigma_x\} = [D]\{\epsilon_x\}$$

shape function matrix evaluated over the surface that the distributed surface traction acts

In 1-D: $[D] = E$

$$\{\hat{d}\} = \begin{Bmatrix}\hat{d}_{1x}\\ \hat{d}_{2x}\end{Bmatrix}$$

$$\pi_p = \frac{A}{2}\int_0^L \{\sigma_x\}^T\{\epsilon_x\} d\hat{x} - \{\hat{d}\}^T \begin{Bmatrix}\hat{f}_{1x}\\ \hat{f}_{2x}\end{Bmatrix} - \iint_{S_1} \{\hat{u}_s\}^T\{\hat{X}_s\}\, dS - \iiint_V \{\hat{u}\}^T\{\hat{X}_b\} dV$$

$$\pi_p = \frac{A}{2}\int_0^L ([D][B]\{\hat{d}\})^T [B]\{\hat{d}\} d\hat{x} - \{\hat{d}\}^T \begin{Bmatrix}\hat{f}_{1x}\\ \hat{f}_{2x}\end{Bmatrix} - \iint_{S_1} ([N_s]\{\hat{d}\})^T\{\hat{X}_s\}\, dS - \iiint_V ([N]\{\hat{d}\})^T\{\hat{X}_b\} dV$$

$$\pi_p = \frac{A}{2}\int_0^L \{\hat{d}\}^T[B]^T[D]^T[B]\{\hat{d}\} d\hat{x} \boxed{- \{\hat{d}\}^T \begin{Bmatrix}\hat{f}_{1x}\\ \hat{f}_{2x}\end{Bmatrix} - \iint_{S_1} \{\hat{d}\}^T[N_s]^T\{\hat{X}_s\}\, dS - \iiint_V \{\hat{d}\}^T[N]^T\{\hat{X}_b\} dV}$$

$$-\{\hat{d}\}^T\left(\begin{Bmatrix}\hat{f}_{1x}\\ \hat{f}_{2x}\end{Bmatrix} + \iint_{S_1} [N_s]^T\{\hat{X}_s\} dS + \iiint_V [N]^T\{\hat{X}_b\} dV\right) = -\{\hat{d}\}^T\{P\}$$

$$\pi_p = \frac{A}{2}\underbrace{\int_0^L}_{\text{constant}} \{\hat{d}\}^T[B]^T[D]^T[B]\{\hat{d}\} d\hat{x} - \{\hat{d}\}^T\{P\} = \frac{AL}{2}\{\hat{d}\}^T[B]^T[D]^T[B]\{\hat{d}\} - \{\hat{d}\}^T\{P\}$$

Definitions

$$\{\hat{f}_s\} \equiv \iint_{S_1} [N_s]^T \{\hat{X}_s\} dS$$ Surface (Traction) Force

$$\{\hat{f}_b\} \equiv \iiint_V [N]^T \{\hat{X}_b\} dV$$ Body Force

Step 3

$$\frac{\partial \pi_p}{\partial \hat{d}_{1x}} = 0 \qquad \frac{\partial \pi_p}{\partial \hat{d}_{2x}} = 0$$

$$\pi_p = \frac{AL}{2} [\hat{d}_{1x} \quad \hat{d}_{2x}] \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} E \left[-\frac{1}{L} \quad \frac{1}{L} \right] \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix} - [\hat{d}_{1x} \quad \hat{d}_{2x}] \begin{Bmatrix} P_{1x} \\ P_{2x} \end{Bmatrix}$$

$$\Rightarrow \pi_p = \frac{AEL}{2L^2} (\hat{d}_{1x}^2 - 2\hat{d}_{1x}\hat{d}_{2x} + \hat{d}_{2x}^2) - \hat{d}_{1x}P_{1x} - \hat{d}_{2x}P_{2x}$$

Bar element stiffness matrix in local coordinates

$$\frac{\partial \pi_p}{\partial \hat{d}_{1x}} = \frac{AE}{L}(\hat{d}_{1x} - \hat{d}_{2x}) - P_{1x} = 0$$

$$\frac{\partial \pi_p}{\partial \hat{d}_{2x}} = \frac{AE}{L}(-\hat{d}_{1x} + \hat{d}_{2x}) - P_{2x} = 0$$

$$\boxed{\begin{Bmatrix} P_{1x} \\ P_{2x} \end{Bmatrix} = \underbrace{\frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{[\hat{k}]} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}}$$

Summary

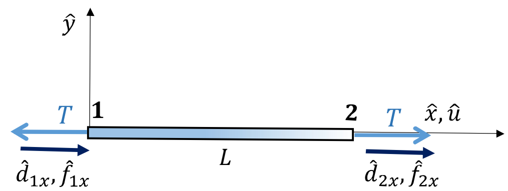
- Potential energy is composed of two parts: internal strain energy and work of external forces
- Internal strain energy takes the effect of stress and strains in the matter
- External forces contain nodal forces, surface forces and body forces.
- The value of P.E. computed by FEM is greater than or equal to reality.
- The stiffness matrix obtained by minimum P.E. approach is the same as the one obtained by direct method.

Further Readings:
Sections 3-10 and its example from "A first course in finite element" by Logan

Galerkin's Residual Method to Derive Bar Element Equations in Local Coordinates

Step 4: Derive the Element Stiffness Matrix and Equations

- Direct Approach
- Minimum Potential Energy
- Galerkin's Method



Weighted Residual Methods

Very good for the situations when we have only the differential equation and boundary conditions available

Weighted Residual Methods

- The methods of weighted residuals applied directly to the differential equation can be used to develop the finite element equations.
- There are a number of other residual methods:
  - collocation,
  - least squares,
  - subdomain

Example:

ODE: $\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} - f(x) = 0$

B.C.s: $u(a) = a', \frac{\partial u}{\partial x}(b) = b'$

Tial Function: ϕ

$\frac{\partial^2 \phi}{\partial x^2} + x\frac{\partial \phi}{\partial x} - f(x) \neq 0$

Differential Equation on a "field function"

Trial Function

$\boxed{\frac{\partial^2 \phi}{\partial x^2} + x\frac{\partial \phi}{\partial x} - f(x) = Residual}$ ⇒R: Residual

Weighted Residual Methods

- We want to have a trial function with the minimum value over the whole region of the element:

$$\iiint_V R\, dV = minimum$$

- In Weighted Residual methods, we require that a weighted value of residual be zero over the whole element:

$$\iiint_V RW\, dV = 0$$

Weighting Function

- In Galerkin's method: $\boxed{\text{W=Shape Functions of the Element}}$

Galerkin's Residual Method

- In an n-DOF element:

$$\boxed{\iiint_V R\, N_i\, dV = 0, \quad (i = 1,2, \dots, n)}$$

A system of n equations

Applies to points within the region of a body without reference to **boundary conditions** such as specified applied loads or displacements.

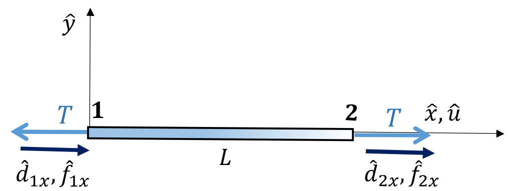
Integration by part

Integrals applicable for the region and its boundary

Galerkin's Residual Method to Derive Bar Element Equations in Local Coordinates

<u>Step 4: Derive the Element Stiffness Matrix and Equations</u>

- Direct Approach
- Minimum Potential Energy
- Galerkin's Method



We had the differential equation for bar element as

$$\frac{d}{d\hat{x}}\left(AE\frac{d\hat{u}}{d\hat{x}}\right) = 0$$

Boundary effects:
$AE\epsilon_x(0) = \hat{f}_{1x}$
$AE\epsilon_x(L) = \hat{f}_{2x}$

Galerkin's Residual Method to Derive Bar Element Equations in Local Coordinates

- We want to derive the equations of FEM for this ODE
- Trial function $\phi = displacement\ function\ obtained\ at\ step\ 2$

$$\phi = [N]\{d\}$$

$$\Rightarrow \frac{d}{d\hat{x}}\left(AE\frac{d\phi}{d\hat{x}}\right) = R$$

Integration by parts is given in general by :

$$\int u\,dv = uv - \int v\,du$$

Boundary Effects
$AE\dfrac{d\phi}{dx}(0) = \hat{f}_{1x}$
$AE\dfrac{d\phi}{dx}(L) = \hat{f}_{2x}$

$$\int_0^L \frac{d}{d\hat{x}}\underbrace{\left(AE\frac{d\phi}{d\hat{x}}\right)}_{v}\underbrace{N_i}_{u}\,d\hat{x} = 0, i = 1,2$$

$$\Rightarrow \int_0^L \frac{d}{d\hat{x}}\left(AE\frac{d\phi}{d\hat{x}}\right)N_i\,d\hat{x} = \underbrace{\left(AE\frac{d\phi}{d\hat{x}}\right)N_i\Big|_0^L}_{\text{Boundary Effects}} - \int_0^L AE\frac{d\phi}{d\hat{x}}\frac{dN_i}{d\hat{x}}\,d\hat{x}$$

Galerkin's Residual Method to Derive Bar Element Equations in Local Coordinates

- $i = 1$ $\quad \int_0^L \frac{d}{d\hat{x}}\left(AE\frac{d\phi}{d\hat{x}}\right)N_1\,d\hat{x} = \underbrace{\left(AE\frac{d\phi}{d\hat{x}}(L)\right)}_{\hat{f}_{2x}}\underbrace{N_1(L)}_{0} - \underbrace{\left(AE\frac{d\phi}{d\hat{x}}(0)\right)}_{\hat{f}_{1x}}\underbrace{N_1(0)}_{1} - \int_0^L AE\frac{d\phi}{d\hat{x}}\frac{dN_1}{d\hat{x}}\,d\hat{x} = 0$
- $i = 2$ $\quad \int_0^L \frac{d}{d\hat{x}}\left(AE\frac{d\phi}{d\hat{x}}\right)N_2\,d\hat{x} = \underbrace{\left(AE\frac{d\phi}{d\hat{x}}(L)\right)}_{\hat{f}_{2x}}\underbrace{N_2(L)}_{1} - \underbrace{\left(AE\frac{d\phi}{d\hat{x}}(0)\right)}_{\hat{f}_{1x}}\underbrace{N_2(0)}_{0} - \int_0^L AE\frac{d\phi}{d\hat{x}}\frac{dN_2}{d\hat{x}}\,d\hat{x} = 0$

Boundary Effects

$AE\frac{d\phi}{dx}(0) = \hat{f}_{1x}$

$AE\frac{d\phi}{dx}(L) = \hat{f}_{2x}$

$-\hat{f}_{1x} - \frac{AE}{L}[1 \quad -1] = 0$

$\hat{f}_{2x} - \frac{AE}{L}[-1 \quad 1] = 0$

Bar element stiffness matrix in local coordinates

$$\begin{Bmatrix}\hat{f}_{1x}\\ \hat{f}_{2x}\end{Bmatrix} = \underbrace{\frac{AE}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}}_{[\hat{k}]}\begin{Bmatrix}\hat{d}_{1x}\\ \hat{d}_{2x}\end{Bmatrix}$$

Summary

- Weighted Residual Methods are good means to solve differential equations numerically
- The basis is to employ a trial function as the solution, which satisfies B.C.s
- The residual is obtained by substituting trial function into D.E.
- We try to get the residual times a weighting function equal to zero over whole volume of the element.
- In Galerkin's method, the weighting function is the same as shape function.
- Integration by part is an important stage in Galerkin's method to introduce B.C.s.

Further Readings:
Sections 3-12 from "A first course in finite element" by Logan

Comparison of FE Solution to exact Solution of Bar Element