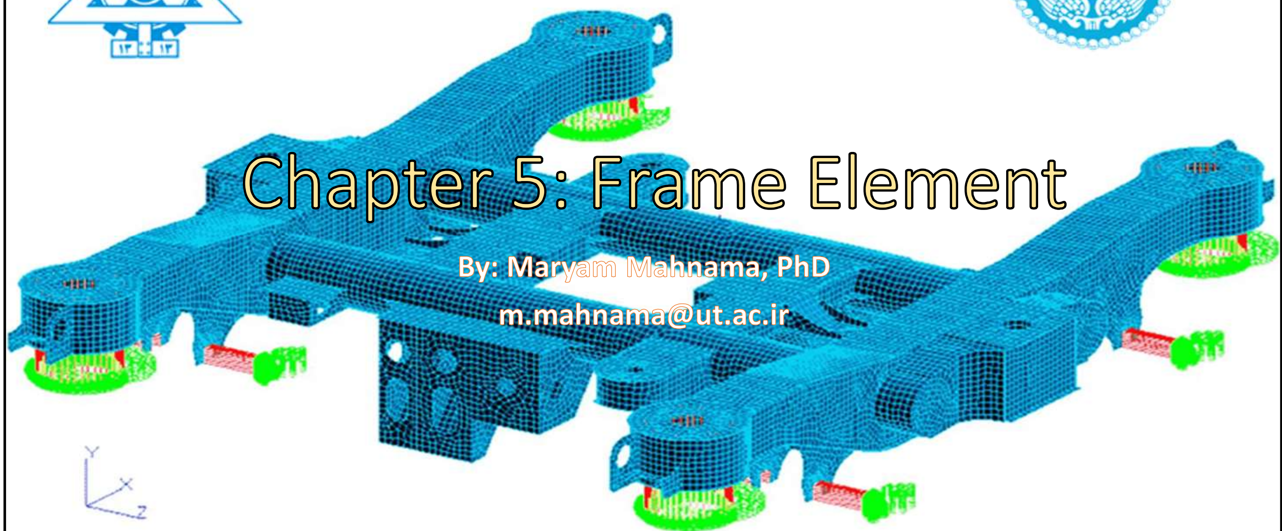
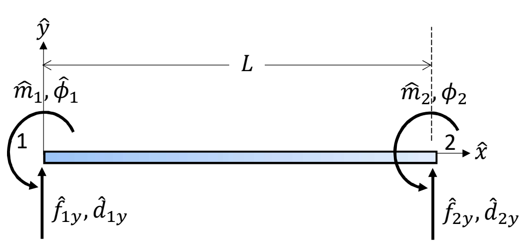


Chapter 5: Frame Element

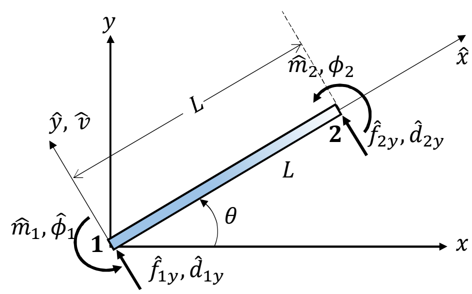
By: Maryam Mahnama, PhD
m.mahnama@ut.ac.ir

Arbitrary-oriented Beam Element



$$\begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2y} \\ \hat{m}_2 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} \hat{d}_{1y} \\ \hat{\phi}_1 \\ \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix}$$

Equations for Beam Element in Local Coordinate



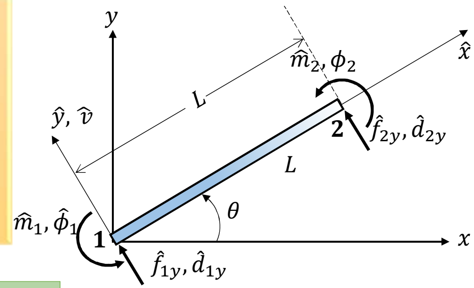
$$\begin{Bmatrix} \hat{d}_x \\ \hat{d}_y \end{Bmatrix} = \begin{bmatrix} C & S \\ -S & C \end{bmatrix} \begin{Bmatrix} d_x \\ d_y \end{Bmatrix}$$

Transformation (rotation) Matrix

$$\boxed{C \equiv \cos\theta, S \equiv \sin\theta}$$

Arbitrary-oriented Beam Element

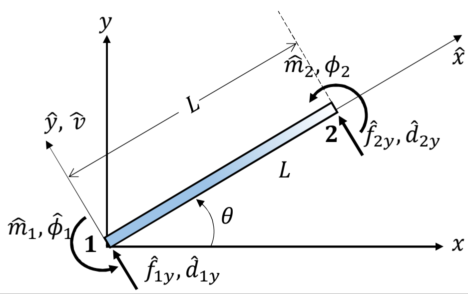
$$\begin{Bmatrix} \hat{d}_{1y} \\ \hat{\phi}_1 \\ \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix} = \underbrace{\begin{bmatrix} -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{[T]\text{: Transformation matrix}} \begin{Bmatrix} d_{1x} \\ d_{1y} \\ \phi_1 \\ d_{2x} \\ d_{2y} \\ \phi_2 \end{Bmatrix}$$

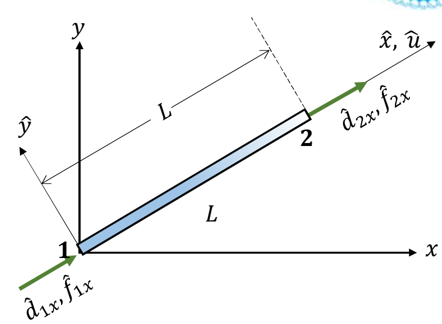


$$\boxed{\boldsymbol{k = T^T \hat{k} T}}$$

$$[K] = \frac{EI}{L^3} \begin{array}{c} \begin{matrix} \quad d_{1x} & \quad d_{1y} & \quad \phi_1 & \quad d_{2x} & \quad d_{2y} & \quad \phi_2 \end{matrix} \\ \begin{bmatrix} 12S^2 & -12CS & -6LS & -12S^2 & -12CS & -6LS \\ -12CS & 12C^2 & 6LC & 12CS & -12C^2 & 6LC \\ -6LS & 6LC & 4L^2 & 6LS & -6LC & 2L^2 \\ -12S^2 & 12CS & 6LS & 12S^2 & -12CS & 6LS \\ -12CS & -12C^2 & -6LC & -12CS & 12C^2 & -6LC \\ -6LS & 6LC & 2L^2 & 6LS & -6LC & 4L^2 \end{bmatrix} \end{array}$$

Adding Axial Effect to the Beam Element





$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ m_1 \\ f_{2x} \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12S^2 & -12CS & -6LS & -12S^2 & -12CS & -6LS \\ -12CS & 12C^2 & 6LC & 12CS & -12C^2 & 6LC \\ -6LS & 6LC & 4L^2 & 6LS & -6LC & 2L^2 \\ -12S^2 & 12CS & 6LS & 12S^2 & -12CS & 6LS \\ -12CS & -12C^2 & -6LC & -12CS & 12C^2 & -6LC \\ -6LS & 6LC & 2L^2 & 6LS & -6LC & 4L^2 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{1y} \\ \phi_1 \\ d_{2x} \\ d_{2y} \\ \phi_2 \end{Bmatrix}$$

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{Bmatrix}$$

Stiffness Matrix for Frame Element

$$[K]=\frac{E}{L}\begin{bmatrix} AC^2+\frac{12I}{L^2}S^2 & \left(A-\frac{12I}{L^2}\right)CS & -\frac{6I}{L}S & -\left(AC^2+\frac{12I}{L^2}S^2\right) & -\left(A-\frac{12I}{L^2}\right)CS & -\frac{6I}{L}S \\ \left(A-\frac{12I}{L^2}\right)CS & AS^2+\frac{12I}{L^2}C^2 & \frac{6I}{L}C & -\left(A-\frac{12I}{L^2}\right)CS & -\left(AS^2+\frac{12I}{L^2}C^2\right) & \frac{6I}{L}C \\ -\frac{6I}{L}S & \frac{6I}{L}S & 4I & \frac{6I}{L}S & -\frac{6I}{L}C & 2I \\ -\left(AC^2+\frac{12I}{L^2}S^2\right) & -\left(A-\frac{12I}{L^2}\right)CS & \frac{6I}{L}S & \left(AC^2+\frac{12I}{L^2}S^2\right) & \left(A-\frac{12I}{L^2}\right)CS & \frac{6I}{L}S \\ -\left(A-\frac{12I}{L^2}\right)CS & -\left(AS^2+\frac{12I}{L^2}C^2\right) & -\frac{6I}{L}C & \left(A-\frac{12I}{L^2}\right)CS & AS^2+\frac{12I}{L^2}C^2 & -\frac{6I}{L}C \\ -\frac{6I}{L}S & \frac{6I}{L}C & 2I & \frac{6I}{L}S & -\frac{6I}{L}C & 4I \end{bmatrix}$$

Rigid Plane Frame

- Definition: A series of beam elements rigidly connected to each other

The original angles made between elements at their joints remain unchanged after the deformation due to applied loads or applied displacements

Moments are transmitted from one element to another at the joints. Hence, moment continuity exists at the rigid joints .

The element centroids, as well as the applied loads, lie in a common plane (for example x-y plane).

Rigid Plane Frame Example

- Solve for the nodal displacements and nodal forces $E = 30 \times 10^6\ psi$
  $A = 10\ in^2$

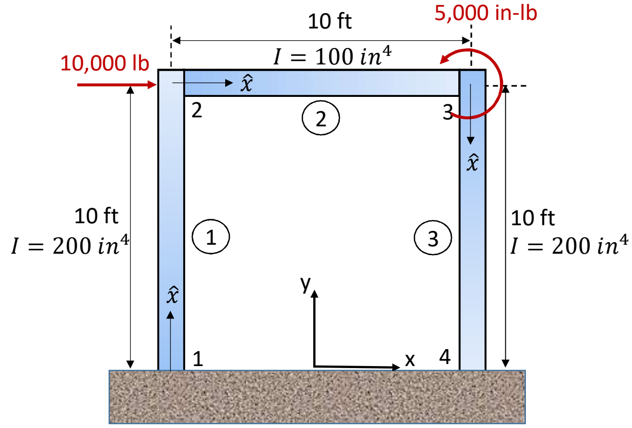
Element #1:

$\theta = 90° \Rightarrow C = 0, \quad S = 1$

$$\frac{12I}{L^2} = \frac{12(200)}{(10 \times 12)^2} = 0.167\ in^2$$

$$\frac{6I}{L} = \frac{6(200)}{10 \times 12} = 10\ in^3$$

$$\frac{E}{L} = \frac{30 \times 10^6}{10 \times 12} = 250{,}000\ lb/in^3$$



Rigid Plane Frame Example

Element #1:

$\theta = -90° \Rightarrow C = 0, \quad S = -1$

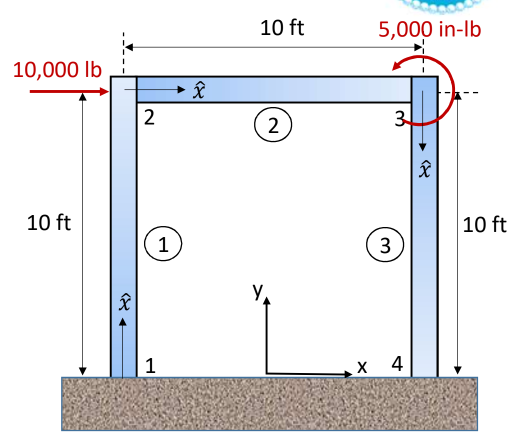
$$\frac{12I}{L^2} = \frac{12(200)}{(10 \times 12)^2} = 0.167\ in^2$$

$$\frac{6I}{L} = \frac{6(200)}{10 \times 12} = 10\ in^3$$

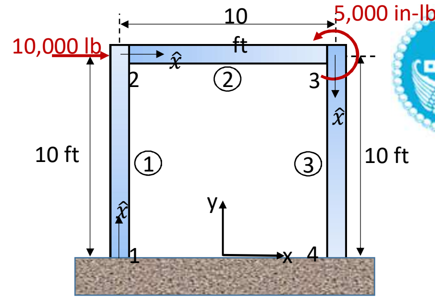
$$\frac{E}{L} = \frac{30 \times 10^6}{10 \times 12} = 250{,}000\ lb/in^3$$

$$\underline{k}^{(1)} = 250{,}000 \begin{array}{c} \begin{array}{cccccc} d_{1x} & d_{1y} & \phi_1 & d_{2x} & d_{2y} & \phi_2 \end{array} \\ \begin{bmatrix} 0.167 & 0 & -10 & -0.167 & 0 & -10 \\ 0 & 10 & 0 & 0 & -10 & 0 \\ -10 & 0 & 800 & 10 & 0 & 400 \\ -0.167 & 0 & 10 & 0.167 & 0 & 10 \\ 0 & -10 & 0 & 0 & 10 & 0 \\ -10 & 0 & 400 & 10 & 0 & 800 \end{bmatrix} \end{array} \frac{\text{lb}}{\text{in.}}$$

$$\underline{k}^{(3)} = 250{,}000 \begin{array}{c} \begin{array}{cccccc} d_{3x} & d_{3y} & \phi_3 & d_{4x} & d_{4y} & \phi_4 \end{array} \\ \begin{bmatrix} 0.167 & 0 & 10 & -0.167 & 0 & 10 \\ 0 & 10 & 0 & 0 & -10 & 0 \\ 10 & 0 & 800 & -10 & 0 & 400 \\ -0.167 & 0 & -10 & 0.167 & 0 & -10 \\ 0 & -10 & 0 & 0 & 10 & 0 \\ 10 & 0 & 400 & -10 & 0 & 800 \end{bmatrix} \end{array} \frac{\text{lb}}{\text{in.}}$$



Rigid Plane Frame Example

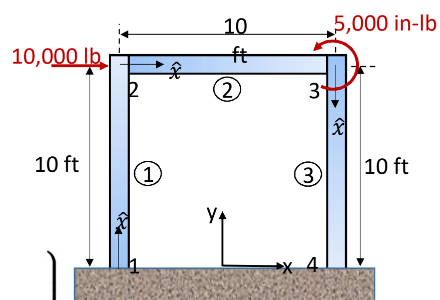


- B.C.s are:

$d_{1x} = d_{1y} = \phi_1 = d_{4x} = d_{4y} = \phi_4 = 0$

$$\begin{Bmatrix} 10{,}000 \\ 0 \\ 0 \\ 0 \\ 0 \\ 5000 \end{Bmatrix} = 250{,}000 \begin{bmatrix} 10.167 & 0 \\ 0 & 10.083 \\ 10 & 5 \\ -10 & 0 \\ 0 & -0.083 \\ 0 & 5 \end{bmatrix} \begin{Bmatrix} d_{2x} \\ d_{2y} \\ \phi_2 \\ d_{3x} \\ d_{3y} \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0.211 \text{ in.} \\ 0.00148 \text{ in.} \\ -0.00153 \text{ rad} \\ 0.209 \text{ in.} \\ -0.00148 \text{ in.} \\ -0.00149 \text{ rad} \end{Bmatrix} \begin{matrix} 0 \\ 5 \\ 200 \\ 10 \\ -5 \\ 1200 \end{matrix}\Bigg] \begin{Bmatrix} d_{2x} \\ d_{2y} \\ \phi_2 \\ d_{3x} \\ d_{3y} \\ \phi_3 \end{Bmatrix}$$

Rigid Plane Frame Example



- In order to comment on the stress, we need to have local forces:
- $\{\hat{d}\} = [T]\{d\} \Rightarrow \{\hat{f}\} = [\hat{k}][T]\{d\}$
- For element #1:

$$\underline{T}\underline{d} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x} = 0 \\ d_{1y} = 0 \\ \phi_1 = 0 \\ d_{2x} = 0.211 \\ d_{2y} = 0.00148 \\ \phi_2 = -0.00153 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.00148 \\ -0.211 \\ -0.00153 \end{Bmatrix}$$

Rigid Plane Frame Example

1

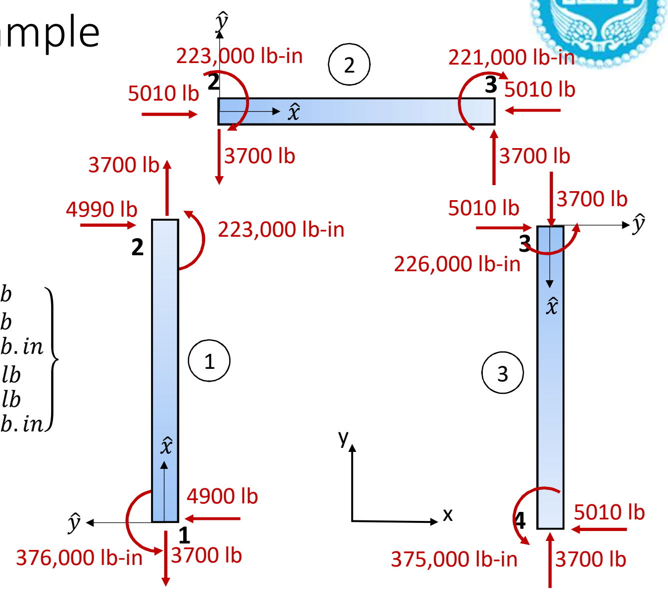
$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2x} \\ \hat{f}_{2y} \\ \hat{m}_2 \end{Bmatrix} = \begin{Bmatrix} -3700\ lb \\ 4990\ lb \\ 376{,}000\ lb.in \\ 3700\ lb \\ -4990\ lb \\ 223{,}000\ lb.in \end{Bmatrix}$$

2

$$\begin{Bmatrix} \hat{f}_{2x} \\ \hat{f}_{2y} \\ \hat{m}_2 \\ \hat{f}_{3x} \\ \hat{f}_{3y} \\ \hat{m}_3 \end{Bmatrix} = \begin{Bmatrix} 5010\ lb \\ -3700\ lb \\ -223{,}000\ lb.in \\ -5010\ lb \\ 3700\ lb \\ -221{,}000\ lb.in \end{Bmatrix}$$

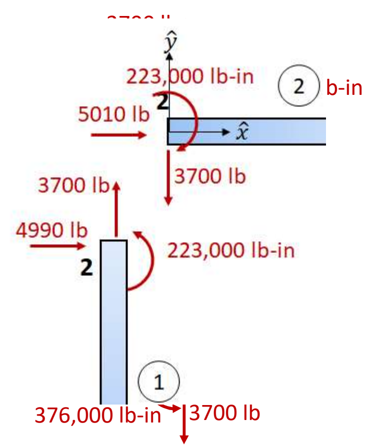
3

$$\begin{Bmatrix} \hat{f}_{3x} \\ \hat{f}_{3y} \\ \hat{m}_3 \\ \hat{f}_{4x} \\ \hat{f}_{4y} \\ \hat{m}_4 \end{Bmatrix} = \begin{Bmatrix} 3700\ lb \\ 5010\ lb \\ 226{,}000\ lb.in \\ -3700\ lb \\ -5010\ lb \\ 375{,}000\ lb.in \end{Bmatrix}$$



Rigid Plane Frame Example

- Equilibrium at each element
- $\sum F_{\hat{x}} = 0 \quad \Rightarrow \quad -3700 + 3700 = 0$
- $\sum F_{\hat{y}} = 0 \quad \Rightarrow \quad 4990 - 4990 = 0$
- $\sum M_2 = 0 \quad \Rightarrow \quad 376{,}000 + 223{,}000 - 4990 \times (120)$
- Equilibrium at each node
- $\sum f_x = F_{2x} \quad \Rightarrow \quad 4990 + 5010 = 10000$
- $\sum f_y = F_{2y} \quad \Rightarrow \quad 3700 - 3700 = 0$
- $\sum m_2 = M_2 \quad \Rightarrow \quad 223{,}000 - 223{,}000 = 0$



Summary

- Frame element is a combination of Bar and Beam elements.
- Stiffness Matrix of Frame Element can be obtained by superposition of that of Bar and Beam element.
- Rigid plane frame is a series of frame elements connected to each other rigidly.

Review of the examples in Chapter 5 of course book