Chapter 8: 2D Elements
Linear-Strain Triangular Element

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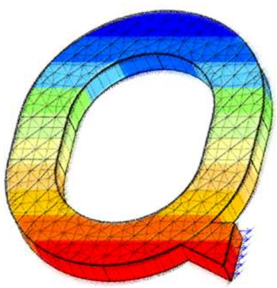
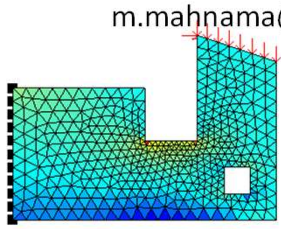




Contents

Derivation of Constant-Strain Triangular Element Stiffness Matrix (Step1)

- **Step 1: Element Type**

The simplest shape in 2D is a triangle

No of nodes: 6

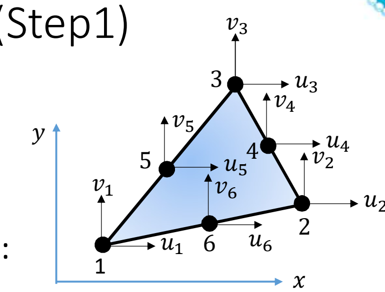
In 2D we have two components for displacement:

- Displacement along x: $u(x, y)$
- Displacement along y: $v(x, y)$

2 DOFs per node

$\Rightarrow$ 12 DOFs in Element:

$$\{d\} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix} \qquad d_1 = \boxed{\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}}$$



Labeling of the nodes should obey a specific standard: Labeling of vertices is always done in counter-clockwise direction. Then the nodes on the base in front of each vertex is labeled.

Derivation of Constant-Strain Triangular Element Stiffness Matrix (Step2)

- Step 2: Select Displacement Functions

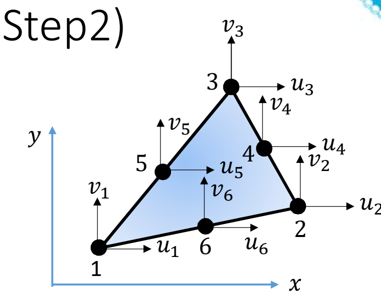
we have two components for displacement:

- Displacement along x: $u(x, y)$ $\Rightarrow$ 6 B.Cs
- Displacement along y: $v(x, y)$ $\Rightarrow$ 6 B.Cs

$$u(x,y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2$$

$$v(x,y) = a_7 + a_8 x + a_9 y + a_{10} x^2 + a_{11} xy + a_{12} y^2$$

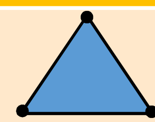
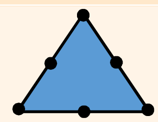
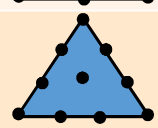
$$\{\psi\} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y & x^2 & xy & y^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{Bmatrix}$$



$$\begin{matrix} & & & 1 & & & \\ & & x & & y & & \\ & x^2 & & xy & & y^2 & \\ x^3 & & x^2y & & xy^2 & & y^3 \end{matrix}$$

Completeness
Symmetry

Relation between Type of Triangular Element & polynomial Coefficients

Terms in Pascal Triangle	Polynomial Degree	No of terms	Triangle Element
1 $x \quad y$	Linear	3	 CST
1 $x \quad y$ $x^2 \quad xy \quad y^2$	Quadrilateral	6	 LST
1 $x \quad y$ $x^2 \quad xy \quad y^2$ $x^3 \quad x^2y \quad xy^2 \quad y^3$	Cubic	10	 QST

Derivation of Constant-Strain Triangular Element Stiffness Matrix (Step2)

$$\{\psi\}=\begin{Bmatrix} u \\ v \end{Bmatrix}=\underbrace{\begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y & x^2 & xy & y^2 \end{bmatrix}}_{[M^*]}\underbrace{\begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{Bmatrix}}_{\{a\}}$$

$$\{\psi\} = [M^*]\,\{a\}$$

$$\underbrace{\begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_6 \\ v_1 \\ \vdots \\ v_5 \\ v_6 \end{Bmatrix}}_{\{d\}}=\underbrace{\begin{bmatrix} 1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & x_2 & y_2 & x_2^2 & x_2y_2 & y_2^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_6 & y_6 & x_6^2 & x_6y_6 & y_6^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_5 & y_5 & x_5^2 & x_5y_5 & y_5^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_6 & y_6 & x_6^2 & x_6y_6 & y_6^2 \end{bmatrix}}_{[X]}\underbrace{\begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_6 \\ a_7 \\ \vdots \\ a_{11} \\ a_{12} \end{Bmatrix}}_{\{a\}}$$

$$\{a\} = [X]^{-1}\{d\}$$

$$\{\psi\} = \boxed{[M^*][X]^{-1}}\{d\}$$

$$\{\psi\} = \boxed{[N]}\{d\}$$

Derivation of Constant-Strain Triangular Element Stiffness Matrix (Step3)

- Step 3: Define Strain/Displacement & Stress/Strain Relationships

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

In this element, the strain is a linear function of x and y. That is why we call this element as "Linear-Strain Triangular Element"

Linear function of x and y

Constant

$$\{\varepsilon\} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 2x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & x & 2y \\ 0 & 0 & 1 & 0 & x & 2y & 0 & 1 & 0 & 2x & y & 0 \end{bmatrix}}_{[M']} \underbrace{\begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{Bmatrix}}_{\{a\}}$$

$$\{\epsilon\} = [M']\{a\}$$

$$\{a\} = [X]^{-1}\{d\}$$

$$\{\epsilon\} = [M'][X]^{-1}\{d\}$$

$$\{\epsilon\} = [B]\{d\}$$

$$\{\sigma\} = [D][B]\{d\}$$

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Derivation of Constant-Strain Triangular Element Stiffness Matrix (Step4)

- Step 4: Derive the Element Stiffness Matrix and Equations

$$[k]_{12\times12} = \iiint_V [B]^T_{3\times12}[D]_{3\times3}[B]_{3\times12}\,dV$$

Linear function of x and y

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 & \beta_4 & 0 & \beta_5 & 0 & \beta_6 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 & 0 & \gamma_4 & 0 & \gamma_5 & 0 & \gamma_6 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 & \gamma_4 & \beta_4 & \gamma_5 & \beta_5 & \gamma_6 & \beta_6 \end{bmatrix}$$

Functions of x and y as well as nodal coordinates

It is very cumbersome to obtain stiffness matrix in explicit form

The integration is best carried out numerically

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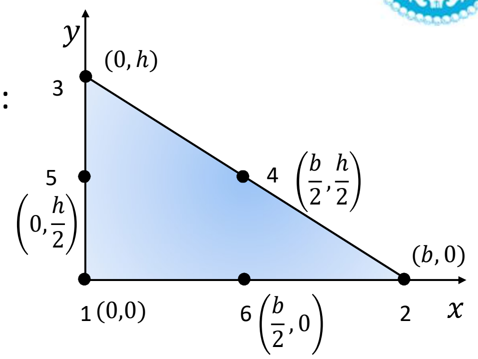
Example Problem

- Obtain the stiffness matrix for LST element:
- First we need to find shape functions:



$$u(x,y) = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2$$

$$u_1 = u(0,0) = a_1$$

$$u_2 = u(b,0) = a_1 + a_2 b + a_4 b^2$$

$$u_3 = u(0,h) = a_1 + a_3 h + a_6 h^2$$

$$u_4 = u\left(\frac{b}{2},\frac{h}{2}\right) = a_1 + a_2\frac{b}{2} + a_3\frac{h}{2} + a_4\left(\frac{b}{2}\right)^2 + a_5\frac{bh}{4} + a_6\left(\frac{h}{2}\right)^2$$

$$u_5 = u\left(0,\frac{h}{2}\right) = a_1 + a_3\frac{h}{2} + a_6\left(\frac{h}{2}\right)^2$$

$$u_6 = u\left(\frac{b}{2},0\right) = a_1 + a_2\frac{b}{2} + a_4\left(\frac{b}{2}\right)^2$$

$$\Rightarrow$$

$$a_1 = u_1 \qquad a_2 = \frac{4u_6 - 3u_1 - u_2}{b} \qquad a_3 = \frac{4u_5 - 3u_1 - u_3}{h}$$

$$a_4 = \frac{2(u_2 - 2u_6 + u_1)}{b^2} \qquad a_5 = \frac{4(u_1 + u_4 - u_5 - u_6)}{bh}$$

$$a_6 = \frac{2(u_3 - 2u_5 + u_1)}{h^2}$$

Example Problem

$$u = u_1 + \left[\frac{4u_6 - 3u_1 - u_2}{b}\right]x + \left[\frac{4u_5 - 3u_1 - u_3}{h}\right]y + \left[\frac{2(u_2 - 2u_6 + u_1)}{b^2}\right]x^2$$
$$+ \left[\frac{4(u_1 + u_4 - u_5 - u_6)}{bh}\right]xy + \left[\frac{2(u_3 - 2u_5 + u_1)}{h^2}\right]y^2$$

$$v = v_1 + \left[\frac{4v_6 - 3v_1 - v_2}{b}\right]x + \left[\frac{4v_5 - 3v_1 - v_3}{h}\right]y + \left[\frac{2(v_2 - 2v_6 + v_1)}{b^2}\right]x^2$$
$$+ \left[\frac{4(v_1 + v_4 - v_5 - v_6)}{bh}\right]xy + \left[\frac{2(v_3 - 2v_5 + v_1)}{h^2}\right]y^2$$

Example Problem

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ \vdots \\ v_6 \end{Bmatrix}$$

$$N_1 = 1 - \frac{3x}{b} - \frac{3y}{h} + \frac{2x^2}{b^2} + \frac{4xy}{bh} + \frac{2y^2}{h^2}$$

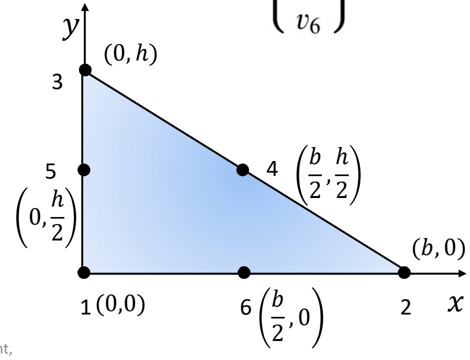
$$N_2 = -\frac{x}{b} + \frac{2x^2}{b^2}$$

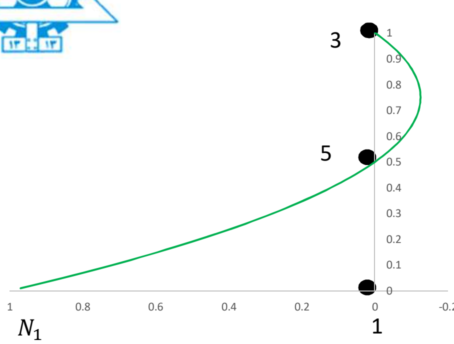
$$N_3 = -\frac{y}{h} + \frac{2y^2}{h^2}$$

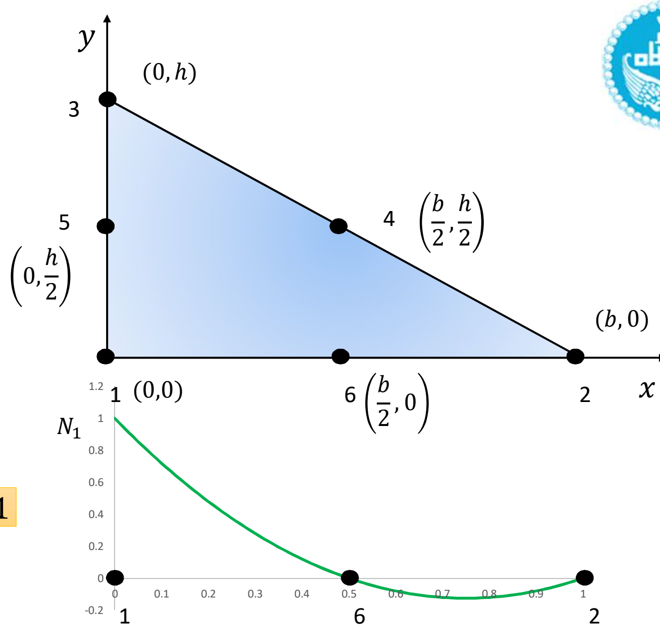
$$N_4 = \frac{4xy}{bh}$$

$$N_5 = \frac{4y}{h} - \frac{4xy}{bh} - \frac{4y^2}{h^2}$$

$$N_6 = \frac{4x}{b} - \frac{4x^2}{b^2} - \frac{4xy}{bh}$$







$$N_1 + N_2 + N_3 + N_4 + N_5 + N_6 = 1$$

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Example Problem

$$N_1 = 1 - \frac{3x}{b} - \frac{3y}{h} + \frac{2x^2}{b^2} + \frac{4xy}{bh} + \frac{2y^2}{h^2}$$
$$N_2 = -\frac{x}{b} + \frac{2x^2}{b^2}$$
$$N_3 = -\frac{y}{h} + \frac{2y^2}{h^2}$$
$$N_4 = \frac{4xy}{bh}$$
$$N_5 = \frac{4y}{h} - \frac{4xy}{bh} - \frac{4y^2}{h^2}$$
$$N_6 = \frac{4x}{b} - \frac{4x^2}{b^2} - \frac{4xy}{bh}$$

$$\beta_1 = (2A)N_{1,x} = -3h + 4hx/b + 4y$$
$$\beta_2 = (2A)N_{2,x} = -h + 4hx/b$$
$$\beta_3 = (2A)N_{3,x} = 0$$
$$\beta_4 = (2A)N_{4,x} = 4y$$
$$\beta_5 = (2A)N_{5,x} = -4y$$
$$\beta_6 = (2A)N_{6,x} = 4h - 8hx/b - 4y$$

$$\gamma_1 = (2A)N_{1,y} = -3b + 4x + 4by/h$$
$$\gamma_2 = (2A)N_{2,y} = 0$$
$$\gamma_3 = (2A)N_{3,y} = -b + 4by/h$$
$$\gamma_4 = (2A)N_{4,y} = 4x$$
$$\gamma_5 = (2A)N_{5,y} = 4b - 4x - 4by/h$$
$$\gamma_6 = (2A)N_{6,y} = -4x$$

Example Problem

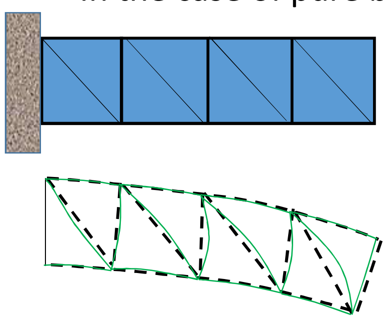
$$\beta_1 = 2A\, N_{1,x} = -3h + 4hx/b + 4y$$
$$\beta_2 = (2A)N_{2,x} = -h + 4hx/b$$
$$\beta_3 = (2A)N_{3,x} = 0$$
$$\beta_4 = (2A)N_{4,x} = 4y$$
$$\beta_5 = (2A)N_{5,x} = -4y$$
$$\beta_6 = (2A)N_{6,x} = 4h - 8hx/b - 4y$$

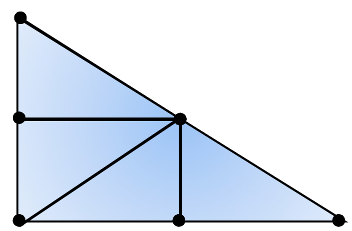
$$\gamma_1 = 2A\, N_{1,y} = -3b + 4x + 4by/h$$
$$\gamma_2 = (2A)N_{2,y} = 0$$
$$\gamma_3 = (2A)N_{3,y} = -b + 4by/h$$
$$\gamma_4 = (2A)N_{4,y} = 4x$$
$$\gamma_5 = (2A)N_{5,y} = 4b - 4x - 4by/h$$
$$\gamma_6 = (2A)N_{6,y} = -4x$$

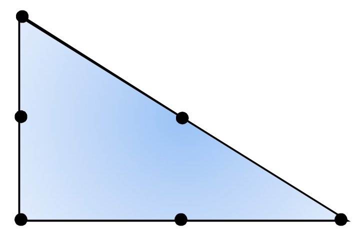
$$\epsilon_x = \frac{1}{2A}[\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 + \beta_4 u_4 + \beta_5 u_5 + \beta_6 u_6]$$
$$\epsilon_y = \frac{1}{2A}[\gamma_1 v_1 + \gamma_2 v_2 + \gamma_3 v_3 + \gamma_4 v_4 + \gamma_5 v_5 + \gamma_6 v_6]$$
$$\gamma_{xy} = \frac{1}{2A}[\gamma_1 u_1 + \beta_1 v_1 + \ \ldots + \gamma_6 u_6 + \beta_6 v_6]$$

Comparison of CST & LST

- For a given number of nodes, a better representation of true stress and displacement is generally obtained using the LST rather than CST.
- In the case of pure bending, LST does not induce fictitious shear.







Summary

- Plane Strain and Plane Stress Formulations can be used for LST Element.
- LST Element was defined and the displacement functions were extracted.
- Stiffness matrix for LST element can be derived using numerical integration.