Chapter 3: Bar (Truss) Element

Comparison of The FEM with Exact Solution

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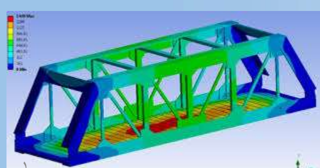


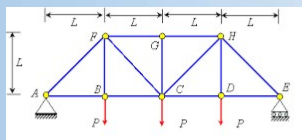


Figure T4.11

Motivation

- We had a differential equation for the displacement in a Bar

$$\frac{d}{d\hat{x}}\left(AE\frac{d\hat{u}}{d\hat{x}}\right) = 0$$

$$\hat{u} = a_0 + a_1\hat{x}$$

2 DOF

Expressions for FEM

Direct Stiffness Method
Minimum Potential Energy
Galerkin's Method

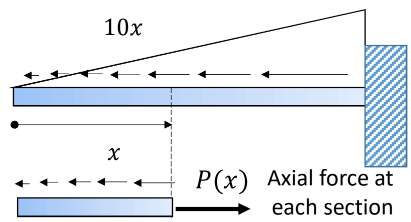
Bar element Stiffness Matrix

Under different loading condition, how precise are the results of FEM? Do they match with exact solution?

A Sample Problem

- A bar is subjected to a linear-varying distributed load:

$$P(x) = \int_0^x 10x'dx' = 5x^2$$



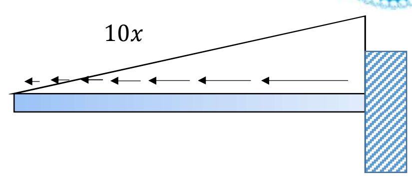
- We can develop the D.E. for this bar:

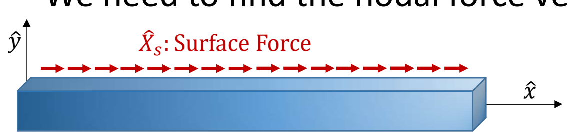
$$AE\frac{du}{dx} = P(x) \qquad u(L) = 0 \quad \Rightarrow \quad u = \frac{1}{AE}\int P(x')dx' \quad \Rightarrow \quad u = \frac{5(x^3 - L^3)}{3AE}$$

Exact solution for the bar

FEM Solution

- We need to find the nodal force vector



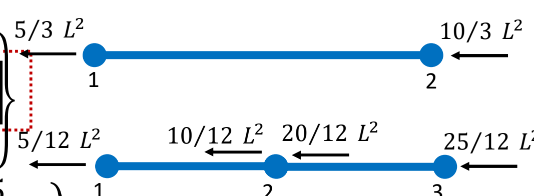


$$\{\hat{f}_s\} = \int_0^{L^{(e)}} [N_s]^T \{\hat{X}_s\} dS$$

Surface (Traction) Force

$$\{\hat{X}_s\} = \{10x\}$$

$$\{\hat{f}_s\}^{(1)} = \int_0^{L^{(e)}} \left[1 - \frac{x}{L^{(e)}} \quad \frac{x}{L^{(e)}}\right]^T \{10x\}\, dx = \begin{Bmatrix} \frac{5}{3}L^2 \\ \frac{10}{3}L^2 \end{Bmatrix} \quad \begin{Bmatrix} \frac{5}{12}L^2 \\ \frac{10}{12}L^2 \end{Bmatrix}$$

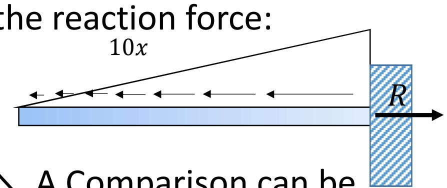


$$\{\hat{f}_s\}^{(2)} = \int_0^{L^{(e)}} \left[1 - \frac{x}{L^{(e)}} \quad \frac{x}{L^{(e)}}\right]^T \{5 + 10x\}\, dx = \begin{Bmatrix} \frac{5}{12}L^2 \\ \frac{10}{12}L^2 \end{Bmatrix}$$

FEM Solution

- We have no body force of nodal force, except the reaction force:

$$\{f\} = \underbrace{\boxed{\{f_s\}}}_{\text{Traction force}} + \underbrace{\boxed{\{P\}}}_{\text{Nodal force}}$$



Displacements can be obtained after imposing boundary conditions

$$\{f\} = [K]\{d\}$$

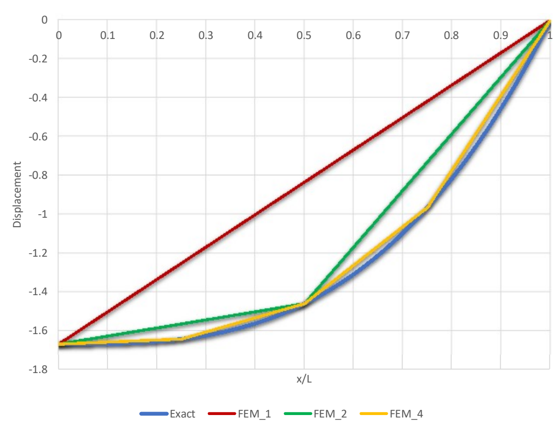
A Comparison can be made between FEM and exact solutions.

Comparison of the Displacement Results

- The displacement in FEM match the exact solution at the node points.

Potential energy is minimized to yield equivalent nodal forces from traction force
⇒ The displacements are exact (only for constant cross-section)

- The values are poor between nodes.
- Increase of # of elements leads to convergence of FEM to exact solution



Comparison of the Stress Results

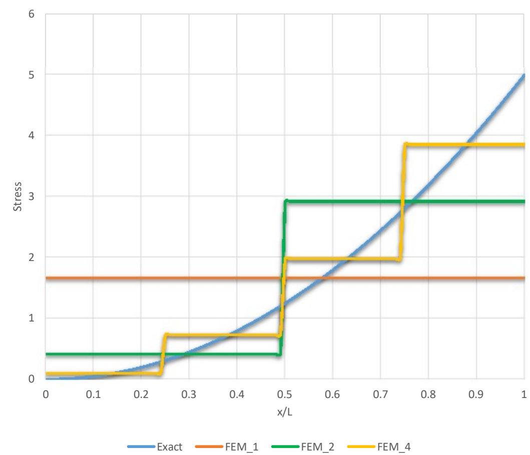
- Stress is derived by $\sigma_x = E\epsilon_x = E\, du/dx$

Displacement is linear ⇒ The stress is constant within the element

- The best approximation for the stress is somewhere in the middle of element.
- The stress is not continuous between elements.

We have no equilibrium between elements

- Increase of # of elements leads to convergence of FEM to exact solution



Summary

- The simple differential equation ($\frac{d}{d\hat{x}}\left(AE\,\frac{d\hat{u}}{d\hat{x}}\right) = 0$) is used to derive the FEM formulation for Bar element.
- Under other forms of loading the D.E. may change and the FEM solution may differ from Exact one.
- Even in such a case, FEM results for displacement matches the exact solution at nodes.
- The results for stress are not similar at nodes, but somewhere within the element, FEM result meets the exact solution.

Further Readings:
Sections 3-11 from “A first course in finite element” by Logan