Chapter 9: Isoparametric Formulation

Development of Bar Element Formulation

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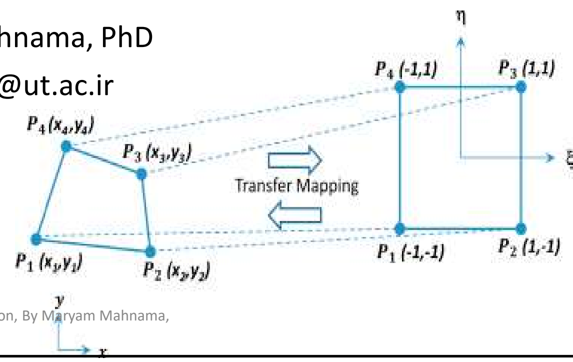


Motivation

- Development of element matrices in terms of a global coordinate system becomes an enormously difficult task (if even possible) except for the simplest of elements such as the CST.
- The isoparametric method may appear somewhat confusing initially, but it will lead to a simple computer program formulation
- It is generally applicable for 2D and 3D stress analysis and for nonstructural problems.
- The isoparametric formulation allows elements to be created that are nonrectangular and have curved sides.
- Furthermore, numerous commercial computer programs have adapted this formulation for their various libraries of elements

Contents of This Chapter

- The isoparametric formulation to develop the simple bar element stiffness matrix.
- Development of the rectangular plane stress element stiffness matrix in terms of a global-coordinate system.
- The isoparametric formulation of the simple quadrilateral element stiffness matrix,

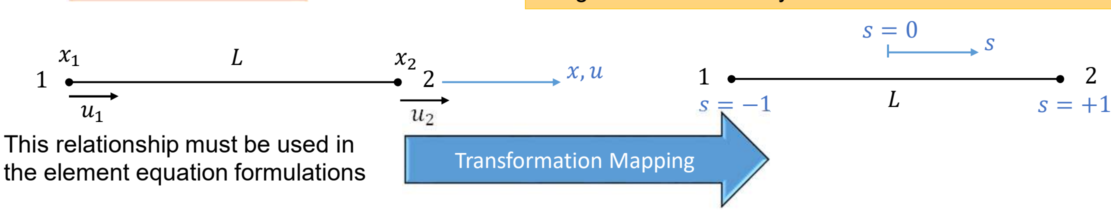
Isoparametric Formulation of the Bar Element Stiffness Matrix

- The term isoparametric is derived from the use of the same shape functions $[N]$ to define the element's geometric shape as are used to define the displacements within the element

$$u = N_1 u_1 + N_2 u_2$$

$$x = N_1 x_1 + N_2 x_2$$

Isoparametric element equations are formulated using a natural (or intrinsic) coordinate system s that is defined by element geometry and not by the element orientation in the global-coordinate system



Development of the Isoparametric Formulation for the Bar Element

- Step 1: Select Element Type
- Step 2: Select a Displacement Function
- Step 3: Define the Strain/Displacement & Stress/Strain Relationships
- Step 4: Derive the Element Stiffness Matrix and Equations

$$x = x_c + \frac{L}{2}s$$

s and x can be related

The natural coordinate s is attached to the element, with the origin located at the center of the element

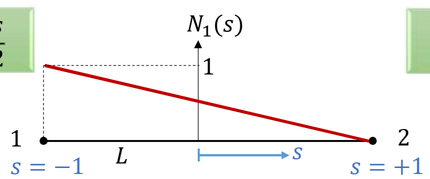
$$x = \frac{x_1 + x_2}{2} + \frac{x_2 - x_1}{2}s$$

$$x = \underbrace{\boxed{\left(\frac{1-s}{2}\right)}}_{N_1} x_1 + \underbrace{\boxed{\left(\frac{1+s}{2}\right)}}_{N_2} x_2$$

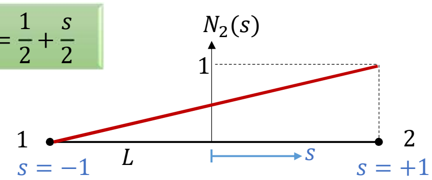
$$x = N_1 x_1 + N_2 x_2$$

Development of the Isoparametric Formulation for the Bar Element (Step1)

$$N_1 = \frac{1}{2} - \frac{s}{2}$$

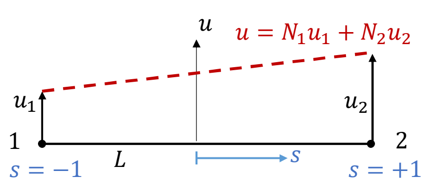


$$N_2 = \frac{1}{2} + \frac{s}{2}$$



$$N_1 + N_2 = 1$$

$$u = N_1 u_1 + N_2 u_2$$

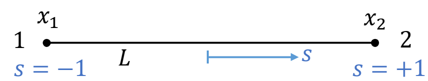


Development of the Isoparametric Formulation for the Bar Element (Step1)

- Another approach to find the shape functions:

We have 2 components to define the geometry of element: x_1, x_2

$$x(s) = a_1 + a_2 s$$



$$x(-1) = a_1 - a_2 = x_1$$

$$x(+1) = a_1 + a_2 = x_2$$

$$\Rightarrow \quad x = \underbrace{\boxed{\left(\frac{1-s}{2}\right)}}_{N_1} x_1 + \underbrace{\boxed{\left(\frac{1+s}{2}\right)}}_{N_2} x_2$$

$$x = [N_1 \quad N_2] \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

Development of the Isoparametric Formulation for the Bar Element (Step2)

- Step 2: Select a Displacement Function
- The displacement function within the bar is now defined by the same shape functions:

$$u = [N_1 \quad N_2] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Since u and x are defined by the same shape functions at the same nodes, the element is called isoparametric

Development of the Isoparametric Formulation for the Bar Element (Step3)

- Step 3: Define the Strain/Displacement & Stress/Strain Relationships

We now want to formulate element matrix [B] to evaluate [k].
⇒ We need the expression for strain

$$\epsilon_x = \frac{du}{dx}$$

$$u = [N_1 \quad N_2]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\epsilon_x = \frac{du}{dx} = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

N_1 and N_2 are functions of s, not x

$$\frac{du}{dx} = \frac{du/ds}{dx/ds}$$

$$\frac{dx}{ds} = \frac{dN_1}{ds}x_1 + \frac{dN_2}{ds}x_2 = \frac{x_2 - x_1}{2} = \frac{L}{2}$$

$$\frac{du}{dx} = \frac{2}{L}\frac{du}{ds}$$

$$\epsilon_x = \frac{2}{L}\begin{bmatrix} \frac{dN_1}{ds} & \frac{dN_2}{ds} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Development of the Isoparametric Formulation for the Bar Element (Step3)

$$\epsilon_x = \frac{2}{L}\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\epsilon_x = \underbrace{\boxed{\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}}}_{[B]}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\sigma_x = E\,\epsilon_x = E[B]\{d\}$$

- Step 4: Derive the Element Stiffness Matrix and Equations

$$[k] = \int_0^L [B]^T E[B]A dx$$

in general, we must transform the coordinate x to s because [B] is, in general, a function of s.

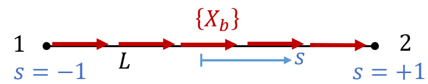
$$\int_0^L f(x)dx = \int_{-1}^{1} f(s)\det(\underbrace{[J]}_{\text{Jacobian Matrix}})\,ds$$

$$\det([J]) = \frac{dx}{ds} = \frac{L}{2}$$

Development of the Isoparametric Formulation for the Bar Element (Step4)

$$[k] = \int_{-1}^{1} [B]^T E [B] A \frac{L}{2} ds = \int_{-1}^{1} \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} E \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} A \frac{L}{2} ds \Rightarrow [k] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

- Body force in isoparametric formulation:



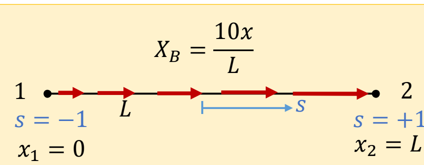
$$\{f_b\} = \int_0^L \boxed{[N]^T}\{X_b\} A\, dx \Rightarrow \{f_b\} = \int_{-1}^{1} [N]^T \{X_b\}\, A \frac{L}{2} ds$$

Function of s, not x

X_B should also be expressed in terms of s

Force Calculation in Isoparametric Formulation (Example)

- Example: we can assume body force as



$$X_B = \frac{10x}{L} = \frac{10}{L}(N_1 x_1 + N_2 x_2) = \frac{5}{L}[(1-s)x_1 + (1+s)x_2]$$

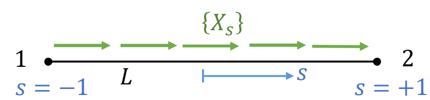
$$X_B = 5(1+s)$$

$$\{f_b\} = \int_{-1}^{1} \begin{Bmatrix} \frac{1-s}{2} \\ \frac{1+s}{2} \end{Bmatrix} 5(1+s) A \frac{L}{2} ds = \frac{5AL}{3}\begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

Force Calculation in Isoparametric Formulation

- Surface force in isoparametric formulation:

$$\{f_s\} = \iint [N_s]^T \{X_s\}\, dS$$



Assuming the cross section is constant and the traction is uniform over the perimeter and along the length of the element,

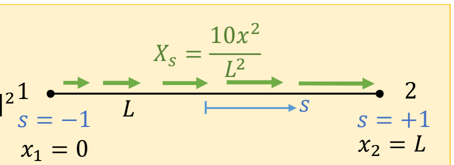
$$\{f_s\} = \int_0^L \boxed{[N_s]^T} \{X_s\}\, dx \quad \Rightarrow \quad \{f_s\} = \int_{-1}^{1} [N_s]^T \{X_s\} \frac{L}{2} ds$$

Function of s, not x

X_B should also be expressed in terms of s

Force Calculation in Isoparametric Formulation (Example)

- Example: we can assume surface force as



$$X_s = \frac{10x^2}{L^2} = \frac{10}{L^2}(N_1x_1 + N_2x_2)^2 = \frac{5}{2L^2}[(1-s)x_1 + (1+s)x_2]^2$$

$$X_s = \frac{5}{2}(1+s)^2$$

$$\{f_s\} = \int_{-1}^{1} \begin{Bmatrix} \frac{1-s}{2} \\ \frac{1+s}{2} \end{Bmatrix} \frac{5}{2}(1+s)^2 A \frac{L}{2} ds = \frac{5AL}{2} \begin{Bmatrix} \frac{1}{3} \\ 1 \end{Bmatrix}$$

Summary

- Plane Strain and Plane Stress Formulations can be used for LST Element.
- LST Element was defined and the displacement functions were extracted.
- Stiffness matrix for LST element can be derived using numerical integration.