Chapter 9: Isoparametric Formulation

Numerical Integration

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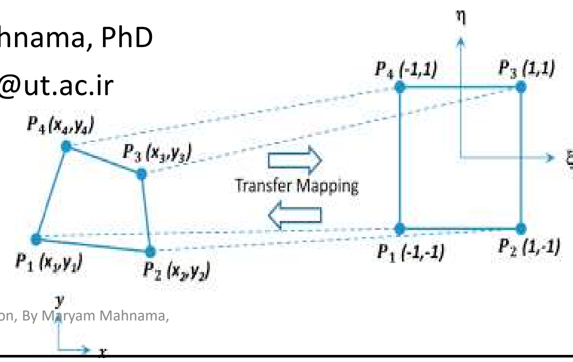


Motivation

In Derivation of equations for Q4 element we faced some definite integrals in the form of:

$$[k] = \int_{-1}^{1}\int_{-1}^{1} \boxed{[B]^T}[D]\boxed{[B]}\, h\, \boxed{|\underline{J}|}\, ds\,dt$$

Functions of s and t

It is hard, if possible, to solve this integral analytically. So, it should be solved numerically.

What is the efficient technique for numerical integration in these cases?

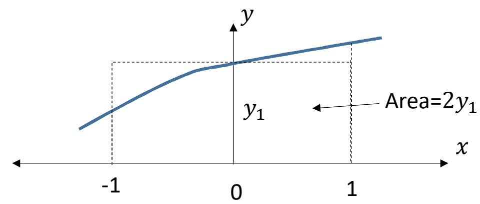
The most useful method for integration in FEM is Gaussian Quadrature method.

Gaussian Quadrature

- How can one evaluate the integral

$$I = \int_{-1}^{1} y\,dx \cong 2y(0)$$

one-point Gaussian quadrature



It is only needed to evaluate the function at one point

- This formula can be generalize as

$$I = \int_{-1}^{1} y\,dx = \sum_{i=1}^{n} \boxed{W_i}\,\boxed{y_i}$$

Weighting

Value of function at sampled point

It is needed to evaluate the function at n points

Gaussian Quadrature

- The rules of Gaussian Quadrature:

Gauss's method chooses the sampling points so that for a given number of points, the best possible accuracy is obtained.

1. Sampling points are located symmetrically with respect to the center of the interval.
2. Symmetrically paired points are given the same weight W_i

Number of Points	Locations, x_i	Associated Weights, W_i
1	$x_1 = 0.000\ldots$	2.000
2	$x_1, x_2 = \pm 0.57735026918962$	1.000
3	$x_1, x_3 = \pm 0.77459666924148$	$\frac{5}{9} = 0.555\ldots$
	$x_2 = 0.000\ldots$	$\frac{8}{9} = 0.888\ldots$
4	$x_1, x_4 = \pm 0.8611363116$	0.3478548451
	$x_2, x_3 = \pm 0.3399810436$	0.6521451549

Gaussian Quadrature

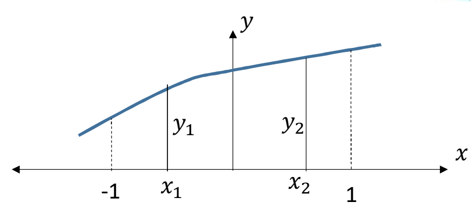
- Note: Gaussian quadrature using n points (Gauss points) is exact if the integrand is a polynomial of degree $2n-1$ or less.

In using n points, we effectively replace the given function $y = f(x)$ by a polynomial of degree $2n-1$. The accuracy of the numerical integration depends on how well the polynomial fits the given curve.

⇒ If the function $f(x)$ is not a polynomial, Gaussian quadrature is inexact, but it becomes more accurate as more Gauss points are used.

Two-Point Gaussian Quadrature

- We want to know how the x_is and W_is are obtained in Gauss's method:



$$I = \int_{-1}^{1} y\,dx = W_1 y_1 + W_2 y_2 = \boxed{W_1} y(\boxed{x_1}) + \boxed{W_2} y(\boxed{x_2})$$

There are four unknowns

$\Rightarrow$ We assume a cubic function for y as $y = C_0 + C_1 x + C_2 x^2 + C_3 x^3$

With four parameters in the two-point formula, we would expect the Gauss formula to exactly predict the area under the curve.

$$A = \int_{-1}^{1} y\,dx = \int_{-1}^{1} C_0 + C_1 x + C_2 x^2 + C_3 x^3\,dx = W_1 y(x_1) + W_2 y(x_2)$$

Two-Point Gaussian Quadrature

$$A = \int_{-1}^{1} y\,dx = \int_{-1}^{1} C_0 + C_1 x + C_2 x^2 + C_3 x^3\,dx = 2C_0 + \frac{2}{3}C_2$$

- According to Gauss's rules $W_1 = W_2 = W$ and $x_1 = -x_2 = a$
- The area predicted by Gauss's formula is

$$A_G = W\,y(a) + W\,y(-a) = 2WC_0 + 2a^2WC_2$$

- If the error, $e = A - A_G$, is to vanish for any C_0 and C_2, we must have

$$e = 2C_0(1-W) + 2C_2\left(\frac{1}{3} - a^2W\right)$$

$$\frac{\partial e}{\partial C_0} = 0 \Rightarrow 2(1-W) = 0 \Rightarrow W = 1$$

$$\frac{\partial e}{\partial C_2} = 0 \Rightarrow 2\left(\frac{1}{3} - a^2W\right) = 0 \Rightarrow a = \frac{1}{\sqrt{3}}$$

$$x_1 = -x_2 = 1/\sqrt{3} = 0.5773$$
$$W_1 = W_2 = 1$$

Example

- We want to find the integral $I = \int_{-1}^{1}[x^2 + \cos(x/2)]dx$ by using 3-point Gaussian quadrature.

Using the table, we have $x_1 = x_3 = \pm 0.77459$ and $x_2 = 0$. Also $W_1 = W_3 = \frac{5}{9}$ and $W_2 = \frac{8}{9}$. So,

$$I = \left[(-0.77459)^2 + \cos\left(-\frac{0.77459}{2}rad\right)\right]\frac{5}{9} + \left[(0)^2 + \cos\left(\frac{0}{2}rad\right)\right]\frac{8}{9} + \left[(0.77459)^2 + \cos\left(\frac{0.77459}{2}rad\right)\right]\frac{5}{9} = 2.585$$

$$I_{exact} = 2.585$$

Gaussian Quadrature in 2D

- In two dimensions, we obtain the quadrature formula by integrating first with respect to one coordinate and then with respect to the other as

$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)ds\,dt = \int_{-1}^{1}\left[\sum_i W_i f(s_i,t)\right]dt = \sum_j W_j\left[\sum_i W_i f(s_i,t_j)\right] = \sum_j\sum_i W_i W_j\, f(s_i,t_j)$$

- In the case of 4 Gauss points (2×2 Gauss points)

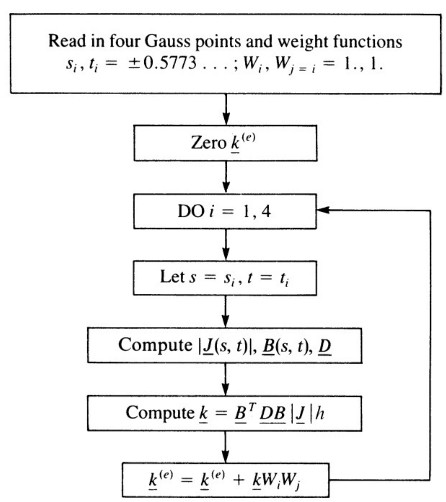
$$I = W_1W_1f(s_1,t_1) + W_1W_2f(s_1,t_2) + W_2W_1f(s_2,t_1) + W_2W_2f(s_2,t_2)$$

Evaluation of the Stiffness Matrix and Stress Matrix by Gaussian Quadrature

- Evaluation of the Stiffness Matrix

$$[k] = \int_{-1}^{1}\int_{-1}^{1} [B(s,t)]^T[D][B(s,t)]\; h\; \left|\underline{J}(s,t)\right| dsdt$$

$$\begin{aligned}\underline{k} = {} & \underline{B}^T(s_1,t_1)\underline{D}\underline{B}(s_1,t_1)|\underline{J}(s_1,t_1)|hW_1W_1 \\ & + \underline{B}^T(s_2,t_2)\underline{D}\underline{B}(s_2,t_2)|\underline{J}(s_2,t_2)|hW_2W_2 \\ & + \underline{B}^T(s_3,t_3)\underline{D}\underline{B}(s_3,t_3)|\underline{J}(s_3,t_3)|hW_3W_3 \\ & + \underline{B}^T(s_4,t_4)\underline{D}\underline{B}(s_4,t_4)|\underline{J}(s_4,t_4)|hW_4W_4\end{aligned}$$



Evaluation of the Stiffness Matrix and Stress Matrix by Gaussian Quadrature

- Evaluation of Element Stresses

$$\boxed{\{\sigma\}} = [D][B(s,t)]\{d\}$$

A function of s and t

There are three ways to compute stress in the element:

1. In practice, the stresses are evaluated at the same Gauss points used to evaluate the stiffness matrix ⇨ For a quadrilateral using 2×2 integration, we get four sets of stress data

2. It is also practical to compute the stresses at $s = t = 0$ in the mid-ponit of the element

3. To evaluate the stresses in all elements at a shared (common) node and then use an average of these element nodal stresses to represent the stress at the node.

Summary

- Gaussian quadrature is the conventional method of numerical integration in FEM.
- Employing this technique, the integration is replaced by summation of the value of integrand at some points times some weightings.
- Employing n Gauss points, the integral can be computed precisely if the integrand is of order 2n-1.
- Stiffness matrix, force vectors and stress values can be computed using Gauss points.