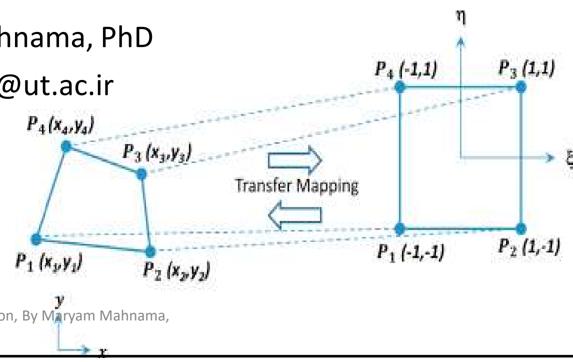
Chapter 9: Isoparametric Formulation

Development of Quadrilateral Element Formulation

By: Maryam Mahnama, PhD

m.mahnama@ut.ac.ir

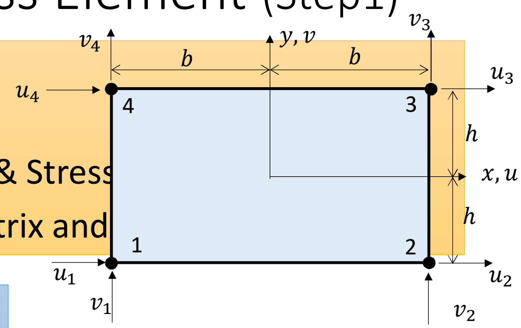


Contents of This Chapter

- The isoparametric formulation to develop the simple bar element stiffness matrix.
- Development of the rectangular plane stress element stiffness matrix in terms of a global-coordinate system.
- The isoparametric formulation of the simple quadrilateral element stiffness matrix,

Development of Equations for Rectangular Plane Stress Element (Step1)

- Step 1: Select Element Type
- Step 2: Select a Displacement Function
- Step 3: Define the Strain/Displacement & Stress
- Step 4: Derive the Element Stiffness Matrix and
  - Displacement along y: $v(x,y)$



2 DOFs per node

$\Rightarrow$ 8 DOFs in Element:

$$\{d\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

Development of Equations for Rectangular Plane Stress Element (Step2)

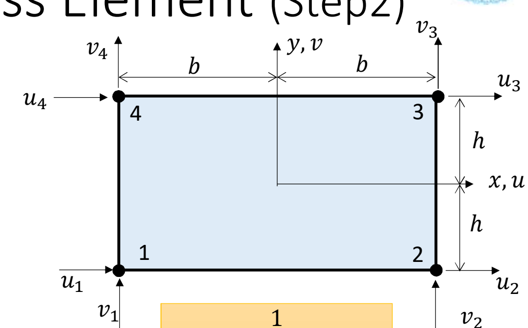
- Step 2: Select a Displacement Function

we have two components for displacement:

- Displacement along x: $u(x,y)$ $\Rightarrow$ 4 B.Cs
- Displacement along y: $v(x,y)$ $\Rightarrow$ 4 B.Cs

$$u(x,y) = a_1 + a_2 x + a_3 y + a_4 xy$$

$$v(x,y) = a_5 + a_6 x + a_7 y + a_8 xy$$



$$\begin{matrix} & & & 1 & & & \\ & & x & & y & & \\ & x^2 & & xy & & y^2 & \\ x^3 & & x^2y & & xy^2 & & y^3 \end{matrix}$$

Completeness

Symmetry

Development of Equations for Rectangular Plane Stress Element (Step2)

- Proceeding to find a_is, we get:

$$u(x,y) = \frac{1}{4bh}[(b-x)(h- \ldots$$

$$+ (b+x)(h+y \ldots$$

$$v(x,y) = \frac{1}{4bh}[(b-x)(h- \ldots$$

$$+ (b+x)(h+y \ldots$$

1

$$\{\psi\} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

Development of Equations for Rectangular Plane Stress Element (Step3)

- Step 3: Define the Strain/Displacement & Stress/Strain Relationships

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

$$\epsilon_x = \frac{1}{4bh}[-(h-y)u_1 + (h-y)u_2 + (h+y)u_3 - (h+y)u_4]$$

$$\epsilon_y = \frac{1}{4bh}[-(b-x)v_1 - (b+x)\, v_2 + (b+x)v_3 + (b-x)v_4]$$

$$\gamma_{xy} = \frac{1}{4bh}[-(b-x)u_1 - (h-y)v_1 + \ \ldots + (b-x)u_4 - (h+y)v_4]$$

$$[B] = \frac{1}{4bh}\begin{bmatrix} -(h-y) & 0 & (h-y) & 0 & (h+y) & 0 & -(h+y) & 0 \\ 0 & -(b-x) & 0 & -(b+x) & 0 & (b+x) & 0 & (b-x) \\ -(b-x) & -(h-y) & -(b+x) & (h-y) & (b+x) & (h+y) & (b-x) & -(h+y) \end{bmatrix}$$

Development of Equations for Rectangular Plane Stress Element (Step4)

- Step 4: Derive the Element Stiffness Matrix and Equations

The stiffness matrix is obtained in global coordinate system.

$$[k] = \int_{-b}^{b}\int_{-h}^{h} [B]^T \boxed{[D]}\boxed{[B]} t\,dy\,dx$$

According to the state of plane stress/plane strian

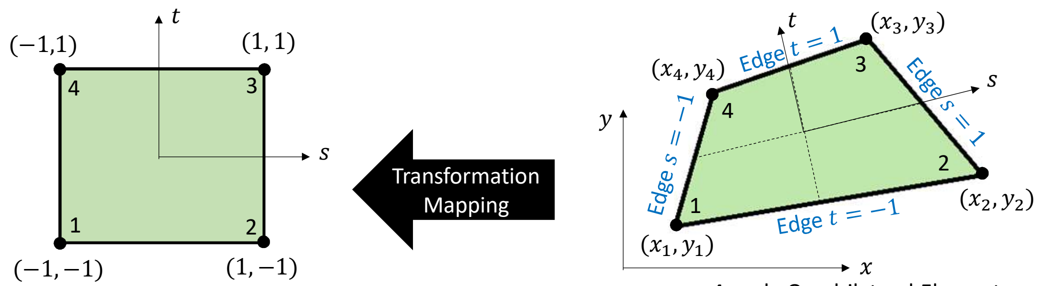
Function of x and y

It is hard to obtain the stiffness matrix analytically, so numerical integration schemes are employed.

Defects of Rectangular Element

- Bilinear rectangle element described in this section also cannot provide pure bending.
- When this element is subjected to pure bending, it also develops false shear strain. (Shear Locking)
- In a pure bending deformation, the bending moment needed to produce the deformation is predicted to be larger than the actual value when modeling with the rectangular element.
- The higher order eight-noded quadratic rectangle has been developed to overcome shear locking.

Isoparametric Formulation of the Quadrilateral Element Stiffness Matrix



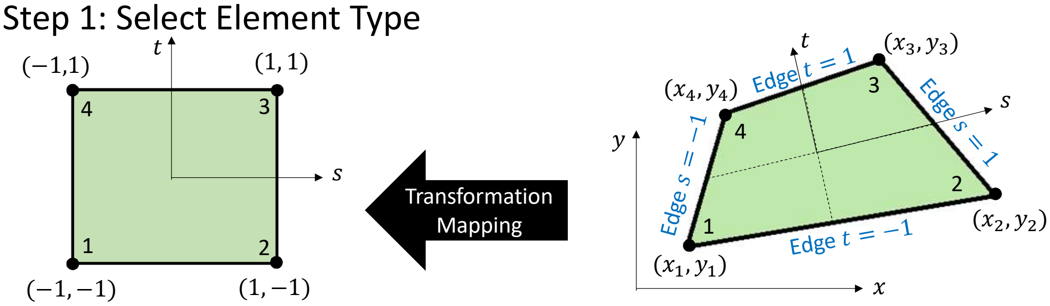
4-node Quadrilateral Element

A rectangular element with $h = b = 1$
The shape functions as well as stiffness matrix are known for this element.

We should obtain some expressions to map (x,y) into (s,t) coordinate system.

Isoparametric Formulation of the Quadrilateral Element Stiffness Matrix (Step1)

- Step 1: Select Element Type



$$x = a_1 + a_2 s + a_3 t + a_4 st$$
$$y = a_5 + a_6 s + a_7 t + a_8 st$$

4 BCs in both directions

$(-1,-1) \Leftarrow (x_1, y_1)$
$(\;\;1,-1) \Leftarrow (x_2, y_2)$
$(\;\;1,-1) \Leftarrow (x_3, y_3)$
$(-1,-1) \Leftarrow (x_4, y_4)$

Isoparametric Formulation of the Quadrilateral Element Stiffness Matrix (Step1)

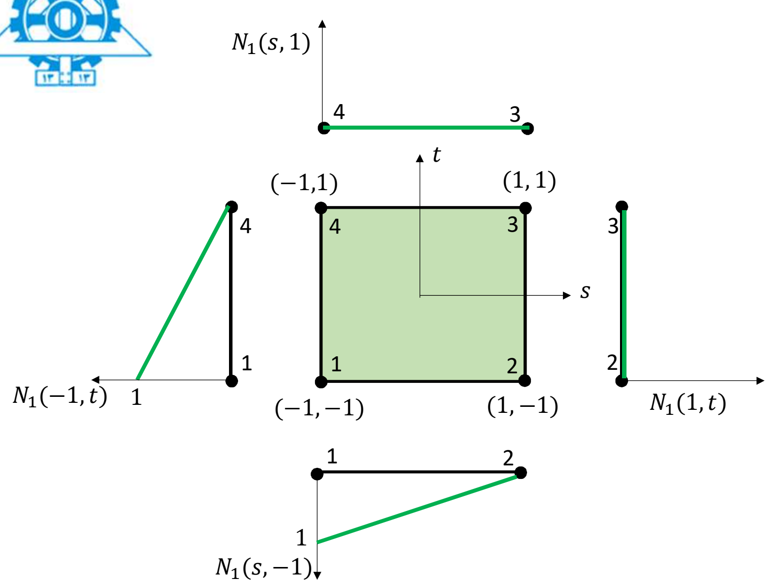
$$x = \frac{1}{4}[(1-s)(1-t)x_1 + (1+s)(1-t)x_2 + (1+s)(1+t)x_3 + (1-s)(1+t)x_4]$$

$$y = \frac{1}{4}[(1-s)(1-t)y_1 + (1+s)(1-t)y_2 + (1+s)(1+t)y_3 + (1-s)(1+t)y_4]$$

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix}$$

$$N_1 = \frac{1}{4}(1-s)(1-t)$$
$$N_2 = \frac{1}{4}(1+s)(1-t)$$
$$N_3 = \frac{1}{4}(1+s)(1+t)$$
$$N_4 = \frac{1}{4}(1-s)(1+t)$$

Bilinear shape function similar to that of rectangular element



$$N_1 = \frac{1}{4}(1-s)(1-t)$$
$$+$$
$$N_2 = \frac{1}{4}(1+s)(1-t)$$
$$+$$
$$N_3 = \frac{1}{4}(1+s)(1+t)$$
$$+$$
$$N_4 = \frac{1}{4}(1-s)(1+t)$$

1

Isoparametric Formulation of the Quadrilateral Element Stiffness Matrix (Step2)

- Step 2: Select a Displacement Function

According to Isoparametric formulation:

$$N_1 = \frac{1}{4}(1-s)(1-t)$$
$$N_2 = \frac{1}{4}(1+s)(1-t)$$
$$N_3 = \frac{1}{4}(1+s)(1+t)$$
$$N_4 = \frac{1}{4}(1-s)(1+t)$$

$$\{\psi\} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

Isoparametric Formulation of the Quadrilateral Element Stiffness Matrix (Step3)

- Step 3: Define the Strain/Displacement & Stress/Strain Relationships

Shape functions are in terms of natural coordinates, not x and y

Strains can be derived from derivatives w.r.t global coordinates

$\Rightarrow$ we need to use chain rule

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

f can be u or v displacement functions, that we need to get their derivatives w.r.t x and y

$$\frac{\partial f}{\partial x} = \frac{\begin{vmatrix} \frac{\partial f}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial f}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix}}{\boxed{\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix}}} \qquad \frac{\partial f}{\partial y} = \frac{\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial f}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial f}{\partial t} \end{vmatrix}}{\boxed{\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix}}}$$

[J]: Jacobian matrix of transformation

Isoparametric Formulation of the Quadrilateral Element Stiffness Matrix (Step3)

$$\{\epsilon\} = [B]\{d\}$$

$$\boxed{\begin{Bmatrix}\epsilon_x\\ \epsilon_y\\ \gamma_{xy}\end{Bmatrix} = \begin{bmatrix}\frac{\partial()}{\partial x} & 0\\ 0 & \frac{\partial()}{\partial y}\\ \frac{\partial()}{\partial y} & \frac{\partial()}{\partial x}\end{bmatrix}\begin{Bmatrix}u\\ v\end{Bmatrix}}$$

$$\frac{\partial()}{\partial x} = \frac{1}{|J|}\left(\frac{\partial y}{\partial t}\frac{\partial()}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial()}{\partial t}\right)$$

$$\frac{\partial()}{\partial y} = \frac{1}{|J|}\left(\frac{\partial x}{\partial s}\frac{\partial()}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial()}{\partial s}\right)$$

$$\begin{Bmatrix}\epsilon_x\\ \epsilon_y\\ \gamma_{xy}\end{Bmatrix} = \boxed{\frac{1}{|J|}\begin{bmatrix}\frac{\partial y}{\partial t}\frac{\partial()}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial()}{\partial t} & 0\\ 0 & \frac{\partial x}{\partial s}\frac{\partial()}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial()}{\partial s}\\ \frac{\partial x}{\partial s}\frac{\partial()}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial()}{\partial s} & \frac{\partial y}{\partial t}\frac{\partial()}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial()}{\partial t}\end{bmatrix}}\boxed{\begin{Bmatrix}u\\ v\end{Bmatrix}}$$

=[N]{d}

[D']: Operator matrix

$$\{\epsilon\} = \underbrace{[D'][N]}_{[B]}\{d\}$$

Isoparametric Formulation of the Quadrilateral Element Stiffness Matrix (Step3)

$$[B(s,t)] = \frac{1}{|J|}[\underline{B_1} \quad \underline{B_2} \quad \underline{B_3} \quad \underline{B_4}]$$

$$\underline{B_i} = \begin{bmatrix}a(N_{i,s}) - b(N_{i,t}) & 0\\ 0 & c(N_{i,t}) - d(N_{i,s})\\ c(N_{i,t}) - d(N_{i,s}) & a(N_{i,s}) - b(N_{i,t})\end{bmatrix}$$

$$a = \tfrac{1}{4}[y_1(s-1) + y_2(-1-s) + y_3(1+s) + y_4(1-s)]$$

$$b = \tfrac{1}{4}[y_1(t-1) + y_2(1-t) + y_3(1+t) + y_4(-1-t)]$$

$$c = \tfrac{1}{4}[x_1(t-1) + x_2(1-t) + x_3(1+t) + x_4(-1-t)]$$

$$d = \tfrac{1}{4}[x_1(s-1) + x_2(-1-s) + x_3(1+s) + x_4(1-s)]$$

$$|J| = \frac{1}{8}\boxed{\{X_c\}^T}\begin{bmatrix}0 & 1-t & t-s & s-1\\ t-1 & 0 & s+1 & -s-t\\ s-t & -s-1 & 0 & t+1\\ 1-s & s+t & -t-1 & 0\end{bmatrix}\boxed{\{Y_c\}}$$

$$[x_1 \quad x_2 \quad x_3 \quad x_4] \qquad [y_1 \quad y_2 \quad y_3 \quad y_4]^T$$

$$\{\sigma\} = [D][B]\{d\}$$

Isoparametric Formulation of the Quadrilateral Element Stiffness Matrix (Step4)

- Step 4: Derive the Element Stiffness Matrix and Equations

The expression for stiffness matrix is

$$[k] = \int\int \boxed{[B]^T}[D]\boxed{[B]}\, h\, dx\, dy \quad \Rightarrow \quad [k] = \int_{-1}^{1}\int_{-1}^{1} \boxed{[B]^T}[D]\boxed{[B]}\, h\, \boxed{|\underline{J}|}\, dsdt$$

Functions of s and t, not x and y

Functions of s and t

It is hard, if possible, to solve this integral analytically. So, it should be solved numerically.

Calculation of Body Force in Isoparametric Formulation

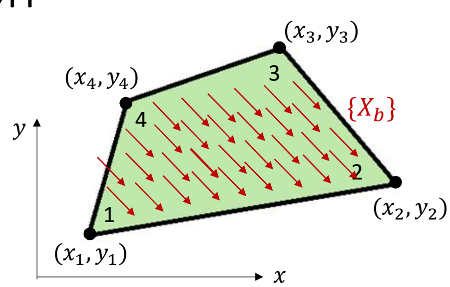
- The general expression for body force is

$$\{f_b\} = \iiint [N]^T\{X_b\}\, dV = \int\int [N]^T\{X_b\} h dxdy$$

$$\Rightarrow \quad \{f_b\} = \int_{-1}^{1}\int_{-1}^{1} [N]^T\{X_b\}\, h\, |\underline{J}|\, dsdt$$

X_b should also be expressed in terms of s and t

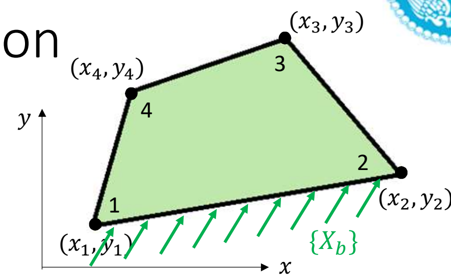
Like the stiffness matrix, the body-force matrix has to be evaluated by numerical integration.



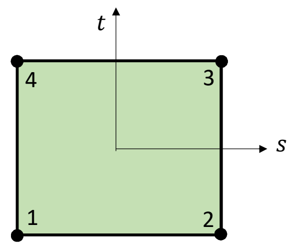
Calculation of Surface Force in Isoparametric Formulation

- The general expression for body force is

$$\{f_s\} = \iint [N_s]^T \{X_s\}\, dS$$



$$\{f_s\} = \int_{-1}^{1} [N_s]^T \{X_s\}\, h \frac{L_{12}}{2}\, ds$$



$$\begin{Bmatrix} f_{s1x} \\ f_{s1y} \\ f_{s2x} \\ f_{s2y} \end{Bmatrix} = \int_{-1}^{1} \boxed{\begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \end{bmatrix}} \begin{Bmatrix} X_{sx} \\ X_{sy} \end{Bmatrix} h \frac{L_{12}}{2}\, ds$$

Evaluated at t=-1

Summary

- Using conventional FEM, expressions for shape functions and stiffness matrix was developed for rectangular element.
- Isoparametric formulation is introduced to map a general quadrilateral 4-node element into a rectangular element in natural coordinate.
- Employing coordinate transformation relations, the integral in global coordinate system can be transformed in natural coordinate.
- Expressions for [k], [B], {f} can be obtained easier by Isoparametric formulation.