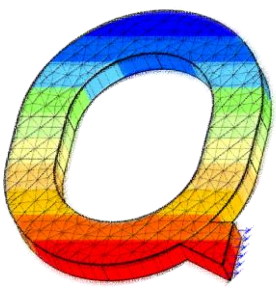
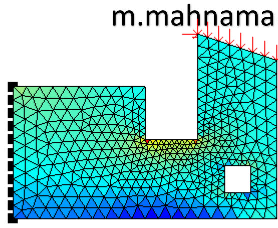
Chapter 6: 2D Elements
Constant-Strain Triangular Element

By: Maryam Mahnama, PhD

m.mahnama@ut.ac.ir

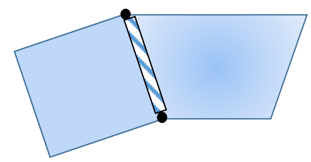




Contents

Introduction to 2D Elements

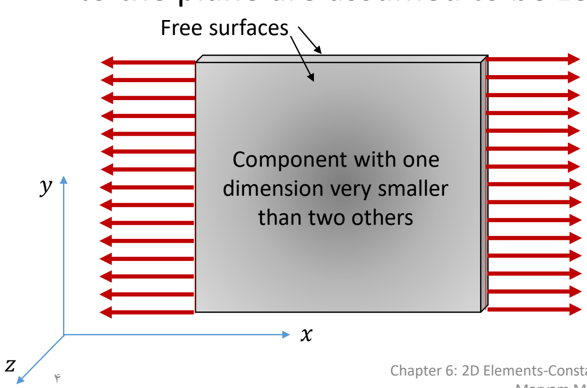
- 1D Elements: only one local coordinate is enough to describe the positions in the element. ⇒ Also called Line Elements
- 2D Elements:
  - Two coordinate are required to describe the positions in the element.
  - Defined by three or more nodes in a 2D plane.
  - Compatibility at common nodes and edges.

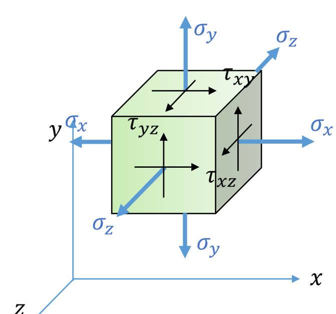


- How can a physical 3D problem be described in 2D?
  - Plane Stress
  - Plane Strain
  - Axisymmetric

Plane stress

- A state of stress in which the normal stress and the shear stresses directed perpendicular to the plane are assumed to be zero:





$$\epsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y$$
$$\epsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x$$
$$\epsilon_z = \frac{1}{E}[\cancelto{0}{\sigma_z} - \nu(\sigma_x$$

$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y)$$
$$\epsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$
$$\epsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$
$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

Plane Strain

- A state of strain in which the strain normal to the x-y plane ϵ_z and the shear strains γ_{xz} and γ_{yz} are assumed to be zero.

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\epsilon_x + \nu\epsilon_y] \quad + \nu\cancelto{0}{\epsilon_z}]$$

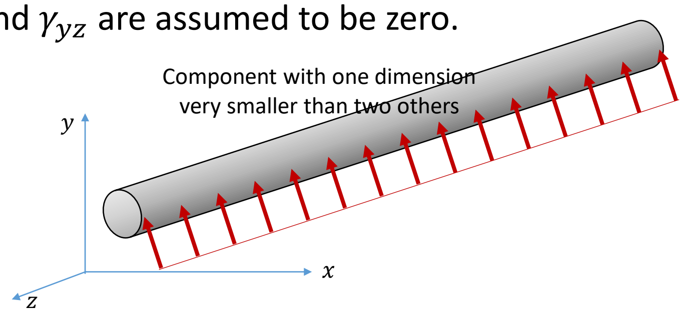
$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}[\nu\epsilon_x + (1-\nu)\epsilon_y] \quad + \nu\cancelto{0}{\epsilon_z}]$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[\nu\epsilon_x + \nu\epsilon_y] \quad \nu)\cancelto{0}{\epsilon_z}]$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\cancelto{0}{\gamma_{xz}} = \frac{\tau_{xz}}{G}$$

$$\cancelto{0}{\gamma_{yz}} = \frac{\tau_{yz}}{G}$$



Plane Stress vs. Plane Strain

Plane Stress:

$$\epsilon_x = \frac{1}{E}(\boxed{\sigma_x} - \nu\sigma_y)$$

$$\epsilon_y = \frac{1}{E}(\boxed{\sigma_y} - \nu\sigma_x)$$

$$\epsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$

$$\gamma_{xy} = \frac{\boxed{\tau_{xy}}}{G}$$

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}$$

$$\sigma_x = \frac{E}{1-\nu^2}(\nu\epsilon_x + \epsilon_y)$$

$$\sigma_y = \frac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_y)$$

$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}$$

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}\nu & 1 & 0\\ 1 & \nu & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}\begin{Bmatrix}\epsilon_x\\ \epsilon_y\\ \gamma_{xy}\end{Bmatrix}$$

[D] for Plane stress

Plane Strain:

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\boxed{\epsilon_x} + \nu\epsilon_y]$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}[\nu\epsilon_x + (1-\nu)\boxed{\epsilon_y}]$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[\nu\epsilon_x + \nu\epsilon_y]$$

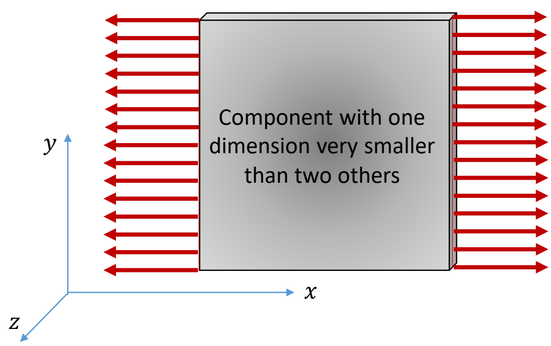
$$\tau_{xy} = G\boxed{\gamma_{xy}}$$

$$\begin{Bmatrix}\epsilon_x\\ \epsilon_y\\ \gamma_{xy}\end{Bmatrix}$$

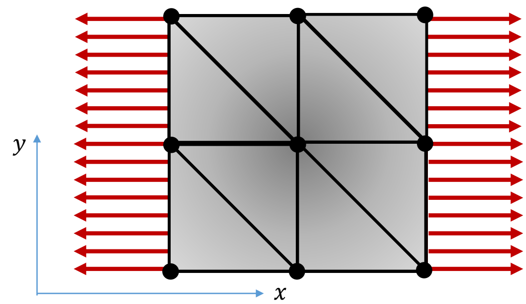
$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}\begin{Bmatrix}\epsilon_x\\ \epsilon_y\\ \gamma_{xy}\end{Bmatrix}$$

[D] for Plane strain

Derivation of Constant-Strain Triangular Element Stiffness Matrix



Plane Stress Assumption



Constant-Strain Triangular Element
(in Plane Stress case)

Derivation of Constant-Strain Triangular Element Stiffness Matrix (Step1)

- **Step 1: Element Type**

The simplest shape in 2D is a triangle

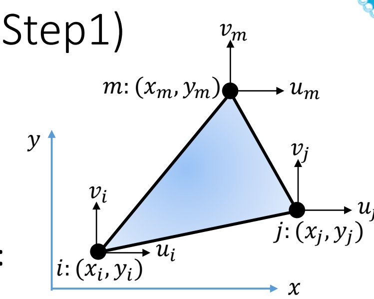
No of nodes: 3

In 2D we have two components for displacement:

- Displacement along x: $u(x,y)$
- Displacement along y: $v(x,y)$

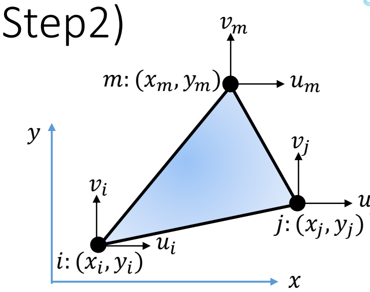
2 DOFs per node

$\Rightarrow$ 6 DOFs in Element: $\{d\} = \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix}$



Labeling of the nodes should obey a specific standard: Labeling is always done in counter-clockwise direction

Derivation of Constant-Strain Triangular Element Stiffness Matrix (Step2)



- Step 2: Select Displacement Functions

we have two components for displacement:

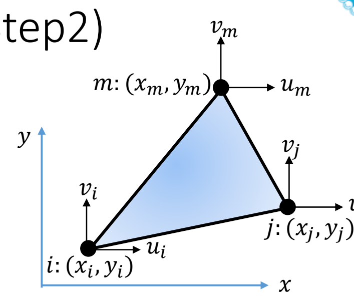
- Displacement along x: $u(x, y)$ ⇨ 3 B.Cs
- Displacement along y: $v(x, y)$ ⇨ 3 B.Cs

$$u(x,y) = a_1 + a_2 x + a_3 y$$

$$v(x,y) = a_4 + a_5 x + a_6 y$$

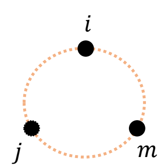
$$\{\psi\} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{Bmatrix} a_1 + a_2 x + a_3 y \\ a_4 + a_5 x + a_6\, y \end{Bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix}$$

Derivation of Constant-Strain Triangular Element Stiffness Matrix (Step2)



- Boundary Conditions are

$$\underbrace{\boxed{\begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix}}}_{\{u\}} = \underbrace{\boxed{\begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{bmatrix}}}_{[x]} \underbrace{\boxed{\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}}}_{\{a\}} \quad \Rightarrow \{u\} = [x]\{a\}$$



$$\begin{Bmatrix} v_i \\ v_j \\ v_m \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{bmatrix} \begin{Bmatrix} a_4 \\ a_5 \\ a_6 \end{Bmatrix}$$

$$\Rightarrow \{a\} = [x]^{-1}\{u\}$$

$$[x]^{-1} = \frac{1}{2A} \begin{bmatrix} \alpha_i & \beta_i & \gamma_i \\ \alpha_j & \beta_j & \gamma_j \\ \alpha_m & \beta_m & \gamma_m \end{bmatrix}$$

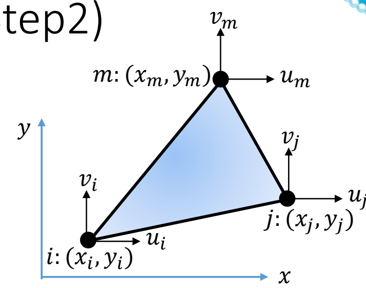
$$\alpha_i = x_j y_m - x_m y_j$$
$$\alpha_j = x_m y_i - x_i y_m$$
$$\alpha_m = x_i y_j - x_j y_i$$

$$2A = \begin{vmatrix} \alpha_i & \beta_i & \gamma_i \\ \alpha_j & \beta_j & \gamma_j \\ \alpha_m & \beta_m & \gamma_m \end{vmatrix}$$

$$\beta_i = y_j - y_m$$
$$\beta_j = y_m - y_i$$
$$\beta_m = y_i - y_j$$

$$\gamma_i = x_m - x_j$$
$$\gamma_j = x_i - x_m$$
$$\gamma_m = x_j - x_i$$

Derivation of Constant-Strain Triangular Element Stiffness Matrix (Step2)



- Boundary Conditions are

$$\begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \Rightarrow \boxed{\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \alpha_i & \beta_i & \gamma_i \\ \alpha_j & \beta_j & \gamma_j \\ \alpha_m & \beta_m & \gamma_m \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix}}$$

$$\begin{Bmatrix} v_i \\ v_j \\ v_m \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{bmatrix} \begin{Bmatrix} a_4 \\ a_5 \\ a_6 \end{Bmatrix} \Rightarrow \boxed{\begin{Bmatrix} a_4 \\ a_5 \\ a_6 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \alpha_i & \beta_i & \gamma_i \\ \alpha_j & \beta_j & \gamma_j \\ \alpha_m & \beta_m & \gamma_m \end{bmatrix} \begin{Bmatrix} v_i \\ v_j \\ v_m \end{Bmatrix}}$$

$$\{u(x,y)\} = \{a_1 + a_2 x + a_3 y\} = [1 \quad x \quad y] \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \boxed{[1 \quad x \quad y] \frac{1}{2A} \begin{bmatrix} \alpha_i & \beta_i & \gamma_i \\ \alpha_j & \beta_j & \gamma_j \\ \alpha_m & \beta_m & \gamma_m \end{bmatrix}} \begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix}$$

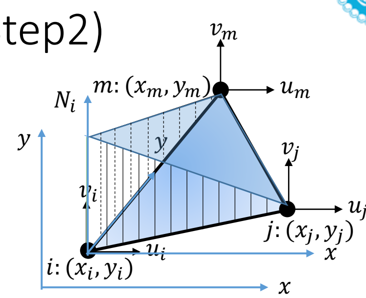
Shape Functions

Derivation of Constant-Strain Triangular Element Stiffness Matrix (Step2)

$$\{u(x,y)\} = \underbrace{\boxed{\frac{1}{2A}(\alpha_i + \beta_i x + \gamma_i y)}}_{N_i(x,y)} u_i + \underbrace{\boxed{\frac{1}{2A}(\alpha_j + \beta_j x + \gamma_j y)}}_{N_j(x,y)} u_j + \underbrace{\boxed{\frac{1}{2A}(\alpha_m + \beta_m x + \gamma_m y)}}_{N_m(x,y)} u_m$$

$$\{v(x,y)\} = \underbrace{\boxed{\frac{1}{2A}(\alpha_i + \beta_i x + \gamma_i y)}}_{N_i(x,y)} v_i + \underbrace{\boxed{\frac{1}{2A}(\alpha_j + \beta_j x + \gamma_j y)}}_{N_j(x,y)} v_j + \underbrace{\boxed{\frac{1}{2A}(\alpha_m + \beta_m x + \gamma_m y)}}_{N_m(x,y)} v_m$$

$$N_i(x_i, y_i) = 1$$
$$N_i(x_j, y_j) = 0$$
$$N_i(x_m, y_m) = 0$$



$$\{\psi\} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \boxed{\begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix}} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix}$$

[N]: Shape Functions Matrix

Derivation of Constant-Strain Triangular Element Stiffness Matrix (Step2)

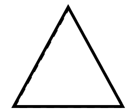
$$N_i(x,y) = \frac{1}{2A}(\alpha_i + \beta_i x + \gamma_i y)$$

$$+$$

$$N_j(x,y) = \frac{1}{2A}(\alpha_j + \beta_j x + \gamma_j y)$$

$$+$$

$$N_m(x,y) = \frac{1}{2A}(\alpha_m + \beta_m x + \gamma_m y)$$



Rigid-Body Displacement

Rigid-Body Rotation

Related to Completeness of Displacement Function

$$\beta_i = y_j - y_m \quad \beta_j = y_m - y_i \quad \beta_m = y_i - y_j$$

$$\gamma_i = x_m - x_j \quad \gamma_j = x_i - x_m \quad \gamma_m = x_j - x_i$$

Completeness: Lower-order terms should not be eliminated in favor of higher-order terms in displacement function.

$$N_i + N_j + N_m = \frac{1}{2A}\left[\underbrace{(\alpha_i + \alpha_j + \alpha_m)}_{2A} + \underbrace{(\beta_i + \beta_j + \beta_m)}_{0}x + \underbrace{(\gamma_i + \gamma_j + \gamma_m)}_{0}y\right] = 1$$

$$\alpha_i = x_j y_m - x_m y_j \quad \alpha_j = x_m y_i - x_i y_m \quad \alpha_m = x_i y_j - x_j y_i$$

$$2A = \begin{vmatrix} \alpha_i & \beta_i & \gamma_i \\ \alpha_j & \beta_j & \gamma_j \\ \alpha_m & \beta_m & \gamma_m \end{vmatrix}$$

Derivation of Constant-Strain Triangular Element Stiffness Matrix (Step3)

- Step 3: Define Strain/Displacement and Stress/Strain Relationships

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(N_i u_i + N_j u_j + N_m u_m)$$
$$\frac{\partial u}{\partial x} = \frac{1}{2A}(\beta_i u_i + \beta_j u_j + \beta_m u_m)$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial x}(N_i u_i + N_j u_j + N_m u_m)$$
$$\frac{\partial u}{\partial y} = \frac{1}{2A}(\gamma_i u_i + \gamma_j u_j + \gamma_m u_m)$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{} \ldots v_j + N_m v_m)$$
$$\ldots v_j + \beta_m v_m)$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial x}(N_i v_i + N_j v_j + N_m v_m)$$
$$\frac{\partial v}{\partial y} = \frac{1}{2A}(\gamma_i v_i + \gamma_j v_j + \gamma_m v_m)$$

$$\{\epsilon\} = \underbrace{\boxed{\frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix}}}_{[B]} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix}$$

Elements of [B] are constant ⇒ Strain does not vary within the element. ⇒ It is called Constant-Strain Triangular (CST) Element

Derivation of Constant-Strain Triangular Element Stiffness Matrix (Step3)

$$\{\epsilon\} = [[B_i] \quad [B_j] \quad [B_m]] \begin{Bmatrix} \{d_i\} \\ \{d_j\} \\ \{d_m\} \end{Bmatrix}$$

$$[B_i] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 \\ 0 & \gamma_i \\ \gamma_i & \beta_i \end{bmatrix} \quad [B_j] = \frac{1}{2A}\begin{bmatrix} \beta_j & 0 \\ 0 & \gamma_j \\ \gamma_j & \beta_j \end{bmatrix} \quad [B_m] = \frac{1}{2A}\begin{bmatrix} \beta_m & 0 \\ 0 & \gamma_m \\ \gamma_m & \beta_m \end{bmatrix}$$

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D]\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = [D][B]\{\epsilon\}$$

Elements of [B] are constant ⇒ Stress does not vary within the CST element.

Derivation of Constant-Strain Triangular Element Stiffness Matrix (Step4)

- Step 4: Derive the Element Stiffness Matrix and Equations

<u>Minimum Potential Energy Approach</u>

Potential Energy is function of element DOFs:

$$\pi_p \equiv \pi_p(u_i, v_i, u_j, v_j, u_m, v_m)$$

$$\pi_p = U + \Omega_b + \Omega_s + \Omega_p$$

$$\Omega_p = -\{d\}^T\{P\}$$

Nodal Force

$$U = \frac{1}{2}\iiint_V \{\epsilon\}^T\{\sigma\}\, dV$$

$[B]\{d\}$

$$\Omega_b = -\iiint_V \{\psi\}^T\{X_b\}\, dV$$

Body Force

$[N]\{d\}$

$$\Omega_s = -\iint_S \{\psi_s\}^T\{X_s\}dS$$

Surface Force

$[N_s]\{d\}$

Derivation of Constant-Strain Triangular Element Stiffness Matrix (Step4)

$$\pi_p = \frac{1}{2}\{d\}^T \iiint_V [B]^T[D][B]\, dV\, \{d\} - \{d\}^T \iiint_V [N]^T\{X_b\}\, dV - \{d\}^T \iint_S [N_s]^T\{X_s\} dS - \{d\}^T\{P\}$$

Definition:

$$\{f\} \equiv \iiint_V [N]^T\{X_b\}\, dV + \iint_S [N_s]^T\{X_s\} dS + \{\mathrm{P}\}$$

$$\pi_p = \frac{1}{2}\{d\}^T \iiint_V [B]^T[D][B]\, dV\, \{d\} - \{d\}^T\, \{f\} \Rightarrow \frac{\partial \pi_p}{\partial \{d\}} = \iiint_V [B]^T[D][B]\, dV\, \{d\} - \{f\} = 0$$

$$\boxed{\iiint_V [B]^T[D][B]\, dV}\{d\} = \{f\}$$

Stiffness Matrix

Derivation of Constant-Strain Triangular Element Stiffness Matrix (Step4)

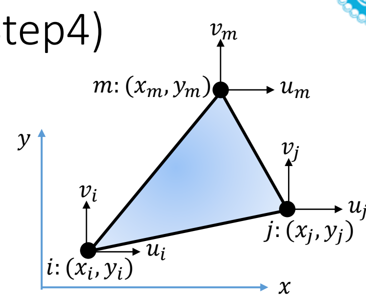
$$[k] = \iiint_V [B]^T[D][B]\, dV$$

$$[k] = t \iint [B]^T[D][B]\, dA$$

Constant

$$[k] = [B]^T[D][B]tA$$

$$[k_{ij}] = [B_i]^T[D][B_j]tA$$



$$[k] = \begin{bmatrix} [k_{ii}]_{2\times2} & [k_{ij}]_{2\times2} & [k_{im}]_{2\times2} \\ [k_{ji}]_{2\times2} & [k_{jj}]_{2\times2} & [k_{jm}]_{2\times2} \\ [k_{mi}]_{2\times2} & [k_{mj}]_{2\times2} & [k_{mm}]_{2\times2} \end{bmatrix}_{6\times6}$$

Assembling Constant-Strain Triangular Element Stiffness Matrices (Step5)

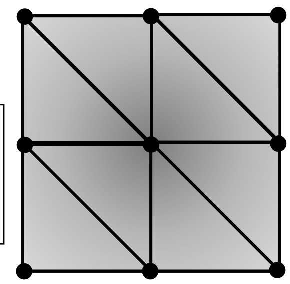
$$\begin{Bmatrix} f_{ix} \\ f_{iy} \\ f_{jx} \\ f_{jy} \\ f_{mx} \\ f_{my} \end{Bmatrix} = [k] \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix}$$

Global Nodal Displacement Vector

Global Nodal Force Vector

The Stiffness Matrix is also in global Coordinate system

Element stiffness Matrices can be assembles without any need for transformation



Summary

- Plane Strain and Plane Stress Formulations were derived.
- CST Element was defined and the displacement functions were extracted.
- Stiffness matrix for CST element was derived.