Chapter 3: Bar (Truss) Element

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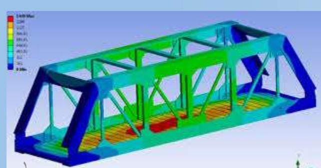


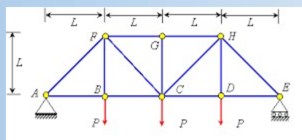
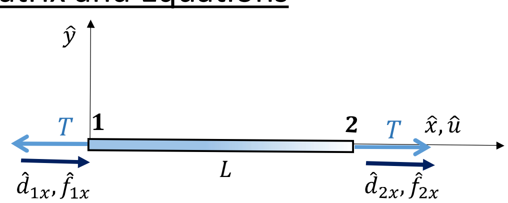


Figure T4.11

Galerkin's Residual Method to Derive Bar Element Equations in Local Coordinates

Step 4: Derive the Element Stiffness Matrix and Equations

- Direct Approach
- Minimum Potential Energy
- Galerkin's Method



Weighted Residual Methods

Very good for the situations when we have only the differential equation and boundary conditions available

Weighted Residual Methods

- The methods of weighted residuals applied directly to the differential equation can be used to develop the finite element equations.
- There are a number of other residual methods:
  - collocation,
  - least squares,
  - subdomain

Example:

ODE: $\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} - f(x) = 0$

B.C.s: $u(a) = a', \frac{\partial u}{\partial x}(b) = b'$

Tial Function: ϕ

$\frac{\partial^2 \phi}{\partial x^2} + x\frac{\partial \phi}{\partial x} - f(x) \neq 0$

Differential Equation on a "field function"

Trial Function

$\frac{\partial^2 \phi}{\partial x^2} + x\frac{\partial \phi}{\partial x} - f(x) = Residual$ ⇒R: Residual

Weighted Residual Methods

- We want to have a trial function with the minimum value over the whole region of the element:

$$\iiint_V R\, dV = minimum$$

- In Weighted Residual methods, we require that a weighted value of residual be zero over the whole element:

$$\iiint_V RW\, dV = 0$$

Weighting Function

- In Galerkin's method: W=Shape Functions of the Element

Galerkin's Residual Method

- In an n-DOF element:

$$\boxed{\iiint_V R\,N_i\,dV = 0, \qquad (i = 1,2,\dots,n)}$$

⇨ A system of n equations

Integration by part

Applies to points within the region of a body without reference to **boundary conditions** such as specified applied loads or displacements.

Integrals applicable for the region and its boundary

Galerkin's Residual Method to Derive Bar Element Equations in Local Coordinates

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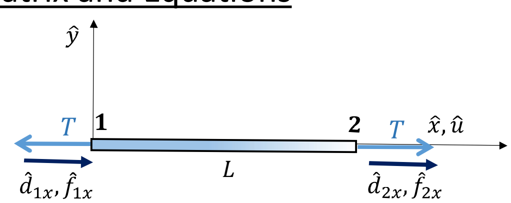

We had the differential equation for bar element as

$$\frac{d}{d\hat{x}}\left(AE\frac{d\hat{u}}{d\hat{x}}\right) = 0$$

Boundary effects:
$AE\epsilon_x(0) = \hat{f}_{1x}$
$AE\epsilon_x(L) = \hat{f}_{2x}$

Galerkin's Residual Method to Derive Bar Element Equations in Local Coordinates

- We want to derive the equations of FEM for this ODE
- Trial function $\phi = displacement\ function\ obtained\ at\ step\ 2$

$\phi = [N]\{d\}$

$$\Rightarrow \frac{d}{d\hat{x}}\left(AE\frac{d\phi}{d\hat{x}}\right) = R$$

Integration by parts is given in general by :

$$\int u\,dv = uv - \int v\,du$$

Boundary Effects

$$AE\frac{d\phi}{dx}(0) = \hat{f}_{1x}$$

$$AE\frac{d\phi}{dx}(L) = \hat{f}_{2x}$$

$$\int_0^L \frac{d}{d\hat{x}}\underbrace{\left(AE\frac{d\phi}{d\hat{x}}\right)}_{v}\underbrace{N_i}_{u}\,d\hat{x} = 0, i = 1,2$$

$$\Rightarrow \int_0^L \frac{d}{d\hat{x}}\left(AE\frac{d\phi}{d\hat{x}}\right)N_i\,d\hat{x} = \underbrace{\left(AE\frac{d\phi}{d\hat{x}}\right)N_i\Big|_0^L}_{\text{Boundary Effects}} - \int_0^L AE\frac{d\phi}{d\hat{x}}\frac{dN_i}{d\hat{x}}\,d\hat{x}$$

Galerkin's Residual Method to Derive Bar Element Equations in Local Coordinates

- $i = 1$

$$\int_0^L \frac{d}{d\hat{x}}\left(AE\frac{d\phi}{d\hat{x}}\right)N_1\,d\hat{x} = \underbrace{\left(AE\frac{d\phi}{d\hat{x}}(L)\right)}_{\hat{f}_{2x}}\underbrace{N_1(L)}_{0} - \underbrace{\left(AE\frac{d\phi}{d\hat{x}}(0)\right)}_{\hat{f}_{1x}}\underbrace{N_1(0)}_{1} - \int_0^L AE\frac{d\phi}{d\hat{x}}\frac{dN_1}{d\hat{x}}\,d\hat{x} = 0$$

- $i = 2$

$$\int_0^L \frac{d}{d\hat{x}}\left(AE\frac{d\phi}{d\hat{x}}\right)N_2\,d\hat{x} = \underbrace{\left(AE\frac{d\phi}{d\hat{x}}(L)\right)}_{\hat{f}_{2x}}\underbrace{N_2(L)}_{1} - \underbrace{\left(AE\frac{d\phi}{d\hat{x}}(0)\right)}_{\hat{f}_{1x}}\underbrace{N_2(0)}_{0} - \int_0^L AE\frac{d\phi}{d\hat{x}}\frac{dN_2}{d\hat{x}}\,d\hat{x} = 0$$

Boundary Effects

$$AE\frac{d\phi}{dx}(0) = \hat{f}_{1x}$$

$$AE\frac{d\phi}{dx}(L) = \hat{f}_{2x}$$

$$-\hat{f}_{1x} - \frac{AE}{L}[1 \quad -1] = 0$$

$$\hat{f}_{2x} - \frac{AE}{L}[-1 \quad 1] = 0$$

Bar element stiffness matrix in local coordinates

$$\Rightarrow \begin{Bmatrix}\hat{f}_{1x}\\ \hat{f}_{2x}\end{Bmatrix} = \underbrace{\frac{AE}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}}_{[\hat{k}]}\begin{Bmatrix}\hat{d}_{1x}\\ \hat{d}_{2x}\end{Bmatrix}$$

Summary

- Weighted Residual Methods are good means to solve differential equations numerically
- The basis is to employ a trial function as the solution, which satisfies B.C.s
- The residual is obtained by substituting trial function into D.E.
- We try to get the residual times a weighting function equal to zero over whole volume of the element.
- In Galerkin's method, the weighting function is the same as shape function.
- Integration by part is an important stage in Galerkin's method to introduce B.C.s.

Further Readings:
Sections 3-12 from "A first course in finite element" by Logan