Chapter 1: Basic Concepts in Finite Element You Need to Know

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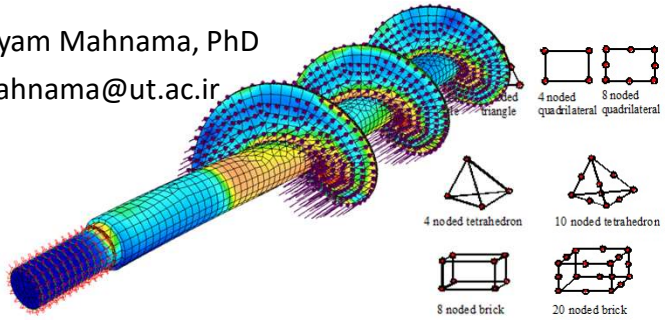


Aims of this lecture

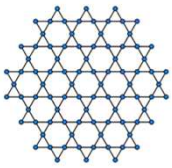
- A brief history of finite element
- Basic definitions
- Different methods to derive finite element equations
- General steps in finite element

History of Finite Element

Origination of FEM: Need for solving complex elasticity and structural analysis problems in civil and aeronautical engineering.

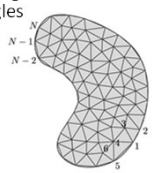
1941

- Alexander Hrennikoff
- Russian Structural Engineer
- Lattice of 1D bar&beam



1942

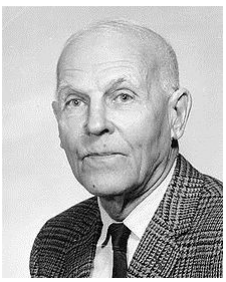
- Richard Courant
- German-American Mathematician
- Solution of stress by variational calculus
- Dividing solid into triangles



1943

- McHenry
- Dividing a continuous solid into bars and beams

Richard Courant



Alexander Hrennikoff

History of Finite Element

Birth of high-speed digital computers

1947

- Levy
- Civil engineer
- Introducing Flexibility (force) method

1953

- Levy
- Civil engineer
- Introducing stiffness (displacement) method

1954

- Argyris & Kelsey
- Development of matrix structural analysis methods using energy principles

1956

- Turner et al.
- Development of stiffness matrices for bar, beam, CST and rectangular elements

1960

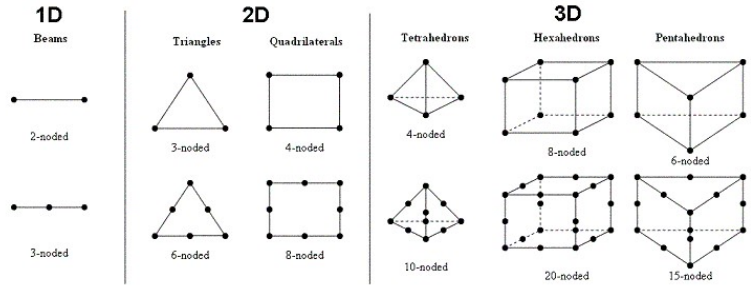
- Clough
- Coined the phrase of "finite element"

• • •

Today, application of finite element method is developed in different fields such as Heat Transfer, Fluid Flow, Electromagnetic, Piezoelectric, Acoustics, Non-linear Dynamics, Modal Analysis, …

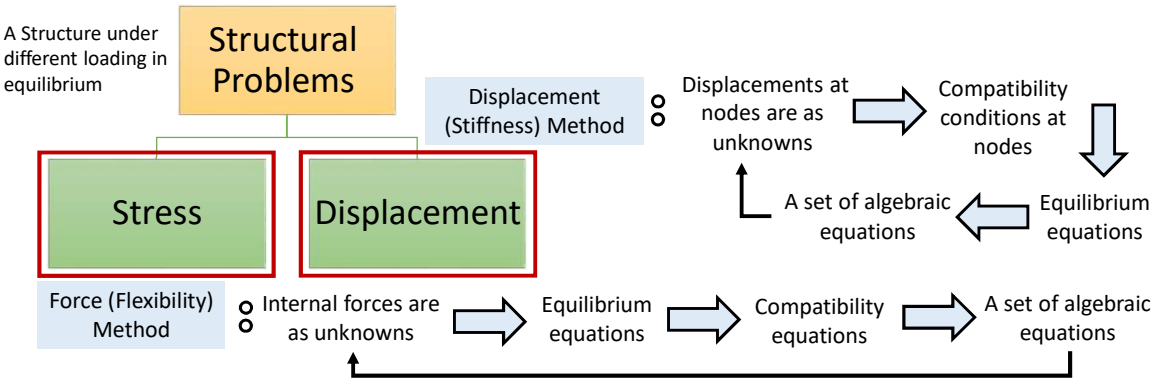
Basic Definitions

- **Elements:** The finite element method involves modeling the structure using small interconnected elements called finite elements.
- **Nodes:** The unknowns are obtained at finite points within the element called nodes.

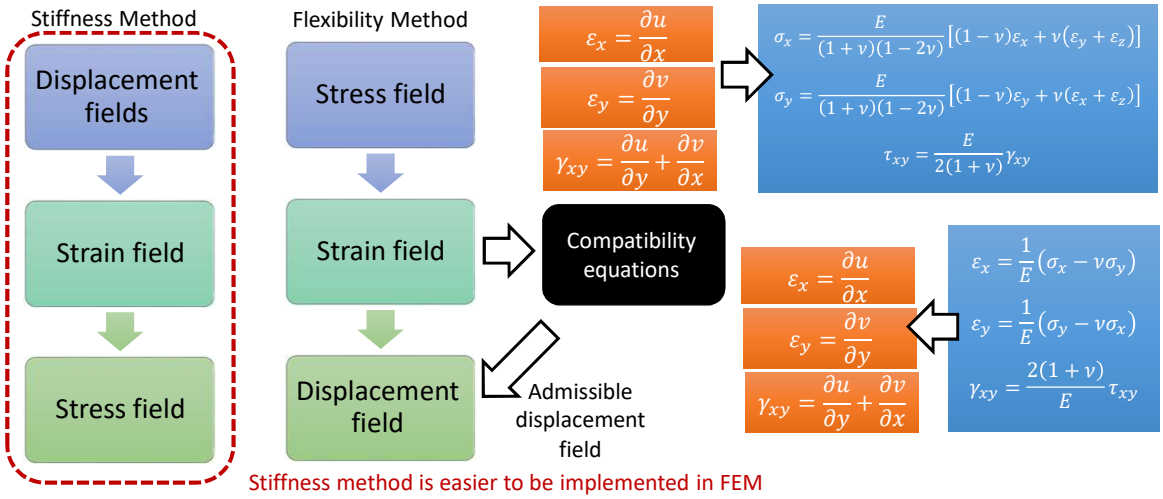


General Steps in Finite Element Method

- There are some steps as guidelines in structural & non-structural problems

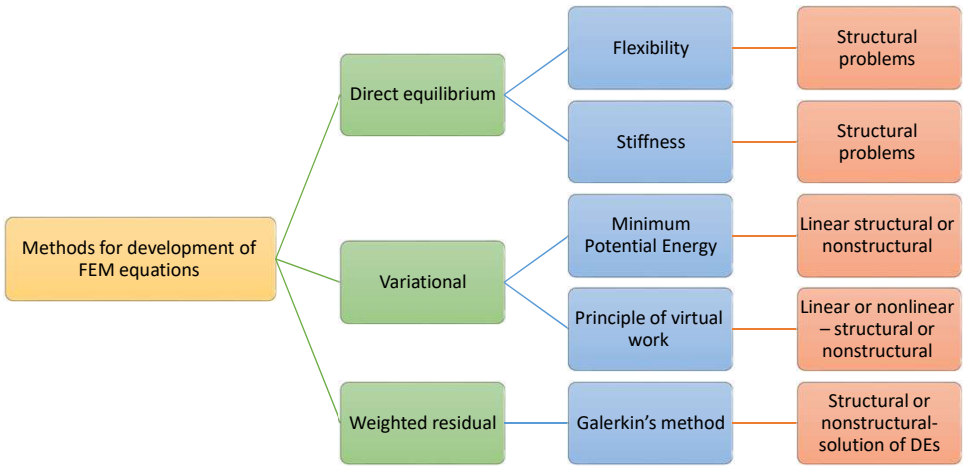


Comparison of Stiffness & Flexibility methods



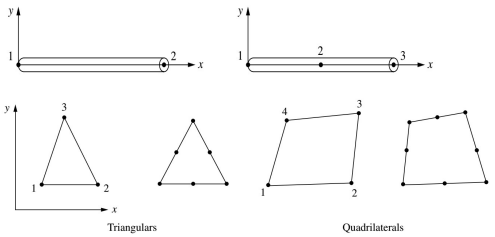
Stiffness method is easier to be implemented in FEM

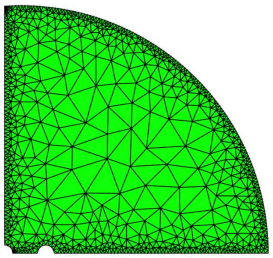
Different methods to derive FE equations



General steps of the FEM

- Step 1: Discretize and Select the Element Types
  - Dividing the body into an equivalent system of finite elements
  - Choosing proper mathematical model to show the physical behavior
  - Size of the elements and variation of element size in different positions
  - Dimension of the element
  - Use of linear or quadratic elements





General steps of the FEM

- Step 2: Select a Displacement Function
  - The function is defined using the nodal values of the element.
  - Polynomials are frequently used functions because they are simple to work with in finite element formulation.
  - In some cases, trigonometric series can also be used.
  - The same general displacement function can be used repeatedly for each element.

Results in several interpolation functions used to find displacement at any point within the element according to the displacements at nodes

General steps of the FEM

- Step 3: Define the Strain-Displacement and Stress-Strain Relationships

$$\varepsilon_x = \frac{\partial u}{\partial x}$$
$$\varepsilon_y = \frac{\partial v}{\partial y}$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Constitutive law

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x + \nu(\varepsilon_y + \varepsilon_z)]$$
$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_y + \nu(\varepsilon_x + \varepsilon_z)]$$
$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}$$

Relationship between stress (force) and displacement is obtained

General steps of the FEM

- Step 4: Derive the Element Stiffness Matrix and Equations

  We should get the equilibrium for element

Direct equilibrium Method

Variational methods

Weighted residual methods

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & \dots & k_{1n} \\ k_{21} & k_{22} & k_{23} & \dots & k_{2n} \\ k_{31} & k_{32} & k_{33} & \dots & k_{3n} \\ \vdots & & & & \vdots \\ k_{n1} & & & \dots & k_{nn} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_n \end{Bmatrix}$$

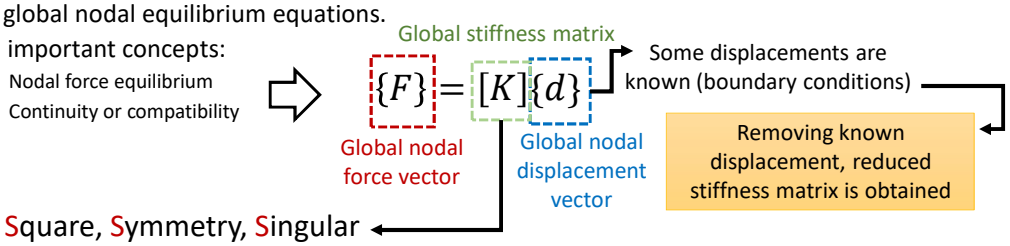
Element nodal force vector

Element stiffness matrix

Element nodal displacement vector

General steps of the FEM

- Step 5: Assemble the Element Equations to Obtain the Global or Total Equations and Introduce Boundary Conditions
  - The individual element nodal equilibrium equations generated in step 4 are assembled into the global nodal equilibrium equations.
  - The important concepts:
    - Nodal force equilibrium
    - Continuity or compatibility



General steps of the FEM

- Step 6: Solve for the Unknown Degrees of Freedom (or Generalized Displacements)

$$\begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & \dots & K_{1n} \\ K_{21} & K_{22} & \dots & K_{2n} \\ \vdots & & & \vdots \\ K_{n1} & K_{n2} & \dots & K_{nn} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{Bmatrix} \Rightarrow \{d\} = [K]^{-1}\{F\}$$

Known forces — Not singular — unknown

General steps of the FEM

- Step 7: Solve for the Element Strains and Stresses

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & \dots & k_{1n} \\ k_{21} & k_{22} & k_{23} & \dots & k_{2n} \\ k_{31} & k_{32} & k_{33} & \dots & k_{3n} \\ \vdots & & & & \vdots \\ k_{n1} & & & \dots & k_{nn} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_n \end{Bmatrix}$$

Element nodal force vector

Element stiffness matrix

Element nodal displacement vector

Summary

- The idea of FEM raised from solving complex elasticity problems in aeronautics
- There are two main approaches to get FE equations: based on force & based on displacement
- Displacement based approaches are more popular
- FE equations can be derived by Direct equilibrium, variational and weighted residual approaches.
- There are 7 main steps to get into FE equations in all types of problems.

For more information on course visit:
Telegram channel: @FEM_Mahnama
Telegram group: FEM open discussion