Chapter 11: Structural Dynamics

Numerical Integration in time

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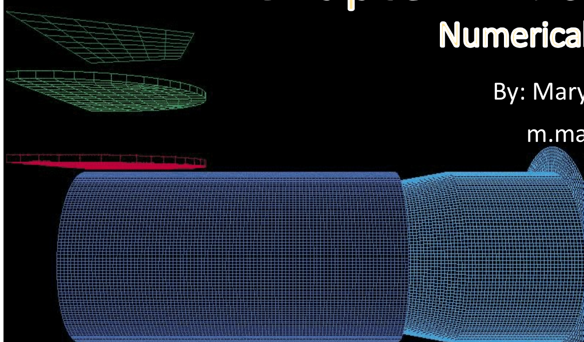
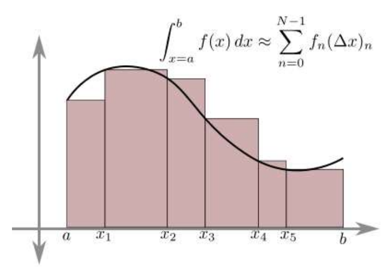
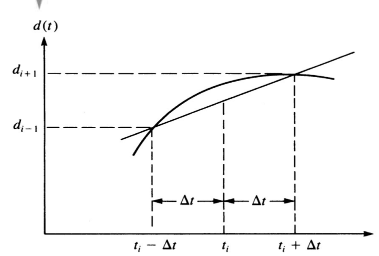


Motivation

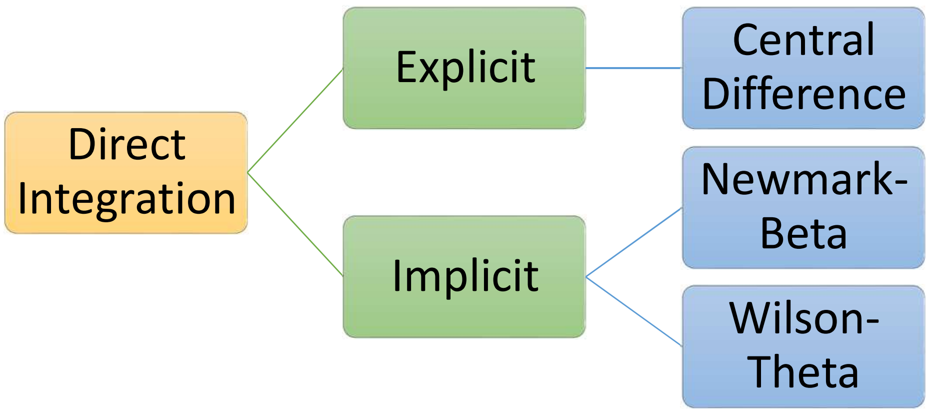
- Numerical Integration in space helped us to get [K].
- In dynamic analysis, displacement changes within time.
- Integration from nodal acceleration gives the evolution of displacement.
- Time integration of acceleration is demanded

The Method is called “Direct Integration”





Direct Integration Methods



Explicit Methods: Central Difference Method

- Based on finite difference expressions in time

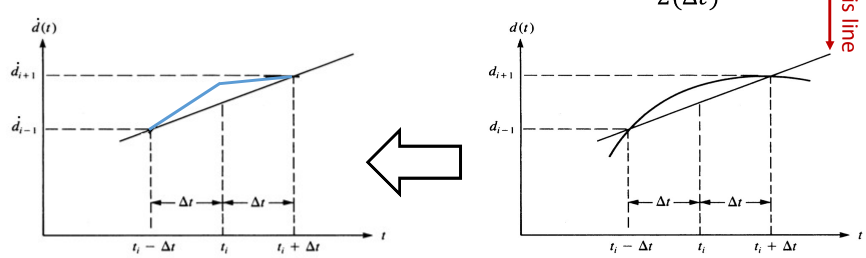
subscript: number of step with time increment of Δt

$$d_i = d(t) \rightarrow d_{i+1} = d(t + \Delta t)$$

$$\dot{d}_i = \frac{d_{i+1} - d_{i-1}}{2(\Delta t)}$$

$$\ddot{d}_i = \frac{\dot{d}_{i+1} - \dot{d}_{i-1}}{2(\Delta t)}$$

Slope of this line



Explicit Methods: Central Difference Method

- Based on finite difference expressions in time

Having initial position and velocity, positions and velocities at every time increments can be obtained.

$$\{\dot{d}_i\} = \frac{\{d_{i+1}\} - \{d_{i-1}\}}{2(\Delta t)}$$

$$\{\ddot{d}_i\} = \frac{\{\dot{d}_{i+1}\} - \{\dot{d}_{i-1}\}}{2(\Delta t)}$$

$$\{\ddot{d}_i\} = \frac{\{d_{i+1}\} - 2\{d_i\} + \{d_{i-1}\}}{(\Delta t)^2}$$

$$\{f^e\} = [\hat{k}]\{\hat{d}\} + [\hat{m}]\{\ddot{\hat{d}}\}$$

$$\{d_{i+1}\} = 2\{d_i\} - \{d_{i-1}\} + (\Delta t)^2\{\ddot{d}_i\}$$

$$\{\ddot{d}_i\} = [M]^{-1}(\{F_i\} - [K]\{d_i\})$$

$$[M]\{d_{i+1}\} = 2[M]\{d_i\} - [M]\{d_{i-1}\} + (\Delta t)^2[M]\{\ddot{d}_i\}$$

$$\{d_{i-1}\} = \{d_i\} - (\Delta t)\{\dot{d}_i\} + \frac{(\Delta t)^2}{2}\{\ddot{d}_i\}$$

$$[M]\{d_{i+1}\} = (\Delta t)^2\{F_i\} + (2[M] - (\Delta t)^2[K])\{d_i\} - [M]\{d_{i-1}\}$$

?

Explicit Methods: Central Difference Method

$\{d_0\},\{\dot{d}_0\}$ & $\{F_i\}$are known

$$\{\ddot{d}_0\} = [M]^{-1}(\{F_0\} - [K]\{d_0\})$$

$$\{d_{-1}\} = \{d_0\} - (\Delta t)\{\dot{d}_0\} + \frac{(\Delta t)^2}{2}\{\ddot{d}_0\}$$

$$\{d_1\} = [M]^{-1}[(\Delta t)^2\{F_0\} + (2[M] - (\Delta t)^2[K])\{d_0\} - [M]\{d_{-1}\}]$$

$$\{d_2\} = [M]^{-1}[(\Delta t)^2\{F_1\} + (2[M] - (\Delta t)^2[K])\{d_1\} - [M]\{d_0\}]$$

$$\{\ddot{d}_1\} = [M]^{-1}(\{F_1\} - [K]\{d_1\})$$

$$\{\dot{d}_1\} = \frac{\{d_2\} - \{d_0\}}{2(\Delta t)}$$

$0 \rightarrow 1$

Implicit Methods: Newmark's (Newmark-Beta) Method

Best approximation for time step: 0.1 of 1/frequency of the desired phenomenon.

$0 < \beta < \frac{1}{4}$, $\gamma = \frac{1}{2}$

$$\{\dot{d}_{i+1}\} = \{\dot{d}_i\} + (\Delta t)[(1-\gamma)\{\ddot{d}_i\} + \gamma\{\ddot{d}_{i+1}\}]$$
$$\{d_{i+1}\} = \{d_i\} + (\Delta t)\{\dot{d}_i\} + (\Delta t)^2[\left(\frac{1}{2} - \beta\right)\{\ddot{d}_i\} + \beta\{\ddot{d}_{i+1}\}]$$

$\beta = 0$, $\gamma = \frac{1}{2}$

$$\{\dot{d}_{i+1}\} = \{\dot{d}_i\} + (\Delta t)[\{\ddot{d}_i\} + \{\ddot{d}_{i+1}\}]/2$$
$$\{d_{i+1}\} = \{d_i\} + (\Delta t)\{\dot{d}_i\} + (\Delta t)^2/2\{\ddot{d}_i\}$$

Central Difference Method

$\beta = \frac{1}{4}$ / $\beta = \frac{1}{6}$, $\gamma = \frac{1}{2}$

$$\{\dot{d}_{i+1}\} = \{\dot{d}_i\} + (\Delta t)[\{\ddot{d}_i\} + \{\ddot{d}_{i+1}\}]/2$$
$$\{d_{i+1}\} = \{d_i\} + (\Delta t)\{\dot{d}_i\} + (\Delta t)^2[\{\ddot{d}_i\} + \{\ddot{d}_{i+1}\}]/4$$
$$\{d_{i+1}\} = \{d_i\} + (\Delta t)\{\dot{d}_i\} + (\Delta t)^2[2\{\ddot{d}_i\} + \{\ddot{d}_{i+1}\}]/6$$

Always stable computational process

Linear acceleration assumption

Displacement and velocities do not become unbounded regardless of the time step chosen.

Implicit Methods: Newmark's (Newmark-Beta) Method

$\{d_0\}, \{\dot{d}_0\}$ & $\{F_i\}$ are known

$$\{\ddot{d}_0\} = [M]^{-1}(\{F_0\} - [K]\{d_0\}) \qquad \{\ddot{d}_1\} = [M]^{-1}(\{F_1\} - [K]\{d_1\})$$

$$[M]\{d_1\} = [M]\{d_0\} + (\Delta t)[M]\{\dot{d}_0\} + (\Delta t)^2[M][\left(\frac{1}{2} - \beta\right)\{\ddot{d}_0\} + \beta\{\ddot{d}_1\}]$$

$$([M] + \beta(\Delta t)^2[K])\{d_1\} = \beta(\Delta t)^2\{F_1\} + [M]\{d_0\} + (\Delta t)[M]\{\dot{d}_0\} + (\Delta t)^2[M]\left(\frac{1}{2} - \beta\right)\{\ddot{d}_0\}$$

$$\underbrace{\left(\frac{[M]}{\beta(\Delta t)^2} + [K]\right)}_{[K']}\{d_1\} = \underbrace{\{F_1\} + \frac{[M]}{\beta(\Delta t)^2}\{d_0\} + \frac{[M]}{\beta(\Delta t)}\{\dot{d}_0\} + \frac{[M]}{\beta}\left(\frac{1}{2} - \beta\right)\{\ddot{d}_0\}}_{\{F'_1\}}$$

$$\{d_1\} = [K']^{-1}\{F'_1\} \qquad \{\dot{d}_1\} = \{\dot{d}_0\} + (\Delta t)[(1-\gamma)\{\ddot{d}_0\} + \gamma\{\ddot{d}_1\}]$$

$0 \to 1$

Implicit Methods: Wilson's (Wilson-Theta) Method

- An extension of the linear acceleration method
- The acceleration is assumed to vary linearly within each time interval from t to $t + \Theta\Delta t$, where $\Theta \geq 1$
- For $\Theta = 1$, the method reduces to linear acceleration method.
- According to references, $\Theta \geq 1.37$ yields unconditional stability.
- In practice $\Theta = 1.4$ is often selected.

Implicit Methods: Wilson's (Wilson-Theta) Method

$$\{\dot{d}_{i+1}\} = \{\dot{d}_i\} + \frac{(\Theta\Delta t)}{2}[\{\ddot{d}_i\} + \{\ddot{d}_{i+1}\}]$$

$$\{d_{i+1}\} = \{d_i\} + (\Theta\Delta t)\{\dot{d}_i\} + \frac{(\Theta\Delta t)^2}{6}(2\{\ddot{d}_i\} + \{\ddot{d}_{i+1}\})$$

$$\{\ddot{d}_{i+1}\} = \frac{6}{(\Theta\Delta t)^2}(\{d_{i+1}\} - \{d_i\}) - \frac{6}{\Theta\Delta t}\{\dot{d}_i\} - 2\{\ddot{d}_i\}$$

$$\{\dot{d}_{i+1}\} = \frac{3}{\Theta\Delta t}(\{d_{i+1}\} - \{d_i\}) - 2\{\dot{d}_i\} - \frac{(\Theta\Delta t)}{2}\{\ddot{d}_i\}$$

Implicit Methods: Wilson's (Wilson-Theta) Method

$\{d_0\},\{\dot{d}_0\}$ & $\{F_i\}$ are known

$$\{\ddot{d}_0\} = [M]^{-1}(\{F_0\} - [K]\{d_0\}) \qquad \{\ddot{d}_1\} = [M]^{-1}(\{F_1\} - [K]\{d_1\})$$

$$[M]\{d_1\} = [M]\{d_0\} + (\Theta\Delta t)[M]\{\dot{d}_0\} + \frac{(\Theta\Delta t)^2}{6}[M][2\{\ddot{d}_0\} + \{\ddot{d}_1\}]$$

$$\left([M] + \frac{(\Theta\Delta t)^2}{6}[K]\right)\{d_1\} = (\Theta\Delta t)^2\{F_1\} + [M]\{d_0\} + (\Theta\Delta t)[M]\{\dot{d}_0\} + \frac{(\Theta\Delta t)^2}{6}[M]\{\ddot{d}_0\}$$

$$\underbrace{\left(\frac{6[M]}{(\Theta\Delta t)^2} + [K]\right)}_{[K']}\{d_1\} = \underbrace{\{F_1\} + \frac{6[M]}{(\Theta\Delta t)^2}\{d_0\} + \frac{6[M]}{(\Theta\Delta t)}\{\dot{d}_0\} + 2[M]\{\ddot{d}_0\}}_{\{F'_1\}}$$

$$\{d_1\} = [K']^{-1}\{F'_1\} \qquad \{\dot{d}_1\} = \{\dot{d}_0\} + (\Theta\Delta t)[\{\ddot{d}_0\} + \{\ddot{d}_1\}]/2$$

$0 \rightarrow 1$

Comparison of Implicit & Explicit Methods

Explicit

- Solution at $t + \Delta t$ is obtained by quantities at t
- Equilibrium eq.s are <u>not</u> satisfied precisely
- Shorter time increments are needed to reach convergence

Implicit

- Solution at $t + \Delta t$ is obtained by quantities at $t + \Delta t$
- Equilibrium eq.s are satisfied **precisely**
- The solution is unconditionally stable

Summary

- In dynamic problems, we need to perform the discretization in both space and time
- Time discretization is used by two main approaches: Implicit and Explicit
- Central Difference method is a sample of Explicit methods.
- Newmak's method and Wilson's method are two examples of implicit schemes.
- Explicit methods are better to be employed in short simulations, while implicit methods are more precise and need more time.