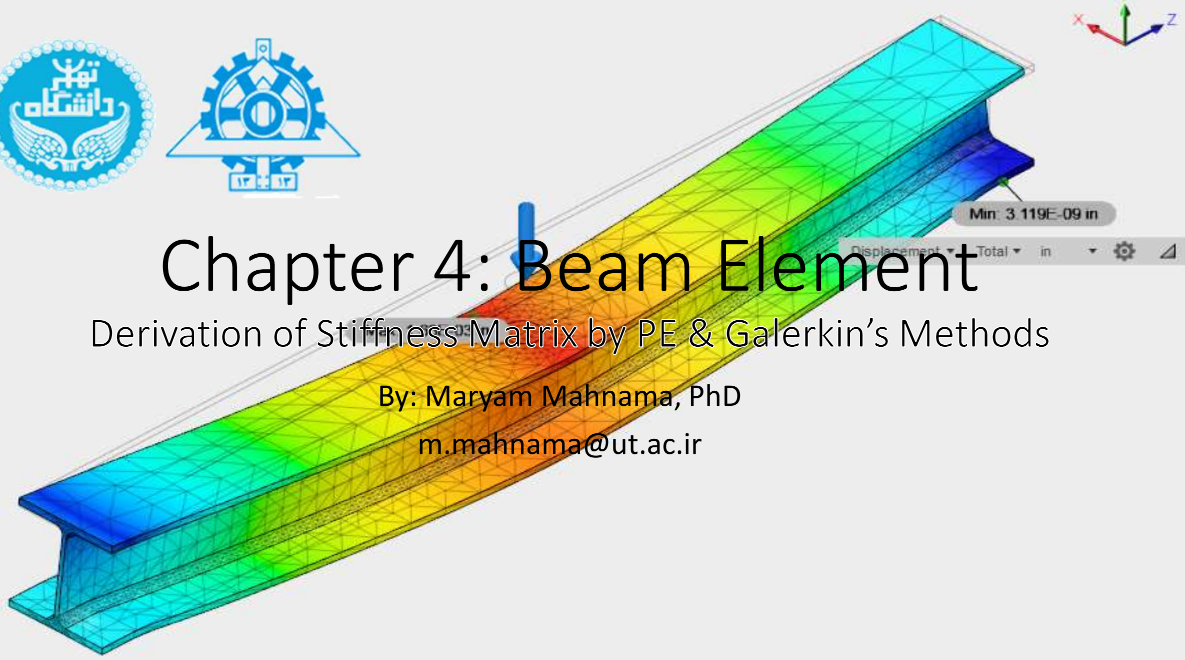

Chapter 4: Beam Element

Derivation of Stiffness Matrix by PE & Galerkin’s Methods

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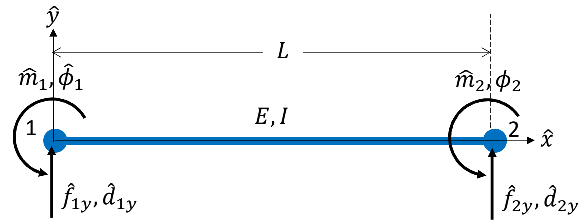
Contents of this lecture

- Derivation of Potential energy expression in beam element
- Minimization of Potential Energy to get element equations
- Derivation of Stiffness Matrix by MPE approach
- Introduction of D.E. of Beam into Galerkin’s Method
- Derivation of Stiffness Matrix by Galerkin’s approach

Minimum Potential Energy to Derive Beam Element Equations in Local Coordinates

Step 4: Derive the Element Stiffness Matrix and Equations

- Direct Approach
- **Minimum Potential Energy**
- Galerkin's Method



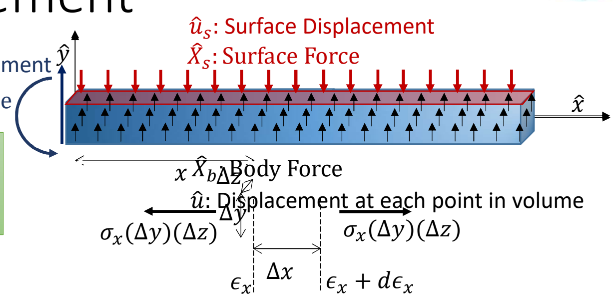
$$\boxed{\pi_p = U + \Omega}$$

internal strain energy	Work of External Forces
The work done by the internal forces through deformations.	The work performed by the external forces during deformations.

Derivation of Potential Energy in Beam Element

$\hat{d}_y, \hat{\phi}$: Nodal Displacement

$\hat{f}_y, \hat{m}$: Nodal Force



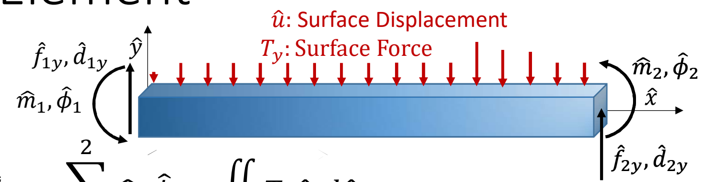
$$dU = \sigma_x(\Delta y)(\Delta z)d\epsilon_x(\Delta x)$$

$$U = \int dU = \frac{1}{2}\iiint_V \sigma_x \epsilon_x \underbrace{dV}_{dAd\hat{x}}$$

For a linear Elastic Material

$$\Omega = -\sum_{i=1}^{M} \hat{f}_{iy}\hat{d}_{iy} - \sum_{i=1}^{M} \hat{m}_i\hat{\phi}_i - \iint_{S_1} \hat{X}_s\hat{u}_s \underbrace{dS}_{bd\hat{x}} - \iiint_V \hat{X}_b\hat{u} \underbrace{dV}_{dAd\hat{x}}$$

Derivation of Potential Energy in Beam Element



$$\pi_p = \frac{1}{2}\iiint_V \sigma_x \epsilon_x dA d\hat{x} - \sum_{i=1}^{2} \hat{f}_{iy}\hat{d}_{iy} - \sum_{i=1}^{2} \hat{m}_i \hat{\phi}_i - \iint_{S_1} T_y \hat{u}\, d\hat{x}$$

$$\boxed{\epsilon_x(\hat{x},\hat{y}) = -\hat{y}\frac{d^2\hat{v}}{d\hat{x}^2}}$$

$$\boxed{\epsilon_x = -\hat{y}\begin{bmatrix} \frac{12\hat{x}-6L}{L^3} & \frac{6\hat{x}L-4L^2}{L^3} & \frac{-12\hat{x}+6L}{L^3} & \frac{6\hat{x}L-4L^2}{L^3} \end{bmatrix}\{\hat{d}\}}$$

$$\boxed{\{\epsilon_x\} = -\hat{y}[B]\{\hat{d}\}} \Rightarrow \boxed{\{\sigma_x\} = -\hat{y}[D][B]\{\hat{d}\}}$$

E (Young's modulus)

Minimization of Potential Energy of Beam Element

$$\pi_p = \frac{1}{2}\iint_A \int_0^L \{\sigma_x\}^T\{\epsilon_x\} d\hat{x}\, dA - \{\hat{d}\}^T\{\hat{P}\} - \int_0^L \{\hat{u}\}^T\{T_y\} b d\hat{x}$$

$$\pi_p = \frac{1}{2}\iint_A \int_0^L y^2\left([D][B]\{\hat{d}\}\right)^T [B]\{\hat{d}\} d\hat{x}\, dA - \{\hat{d}\}^T\{\hat{P}\} - \int_0^L \{\hat{d}\}^T[N]^T\{T_y\} b d\hat{x}$$

$$\boxed{\iint_A y^2\, dA = I}$$

$$\pi_p = \frac{EI}{2}\int_0^L \{\hat{d}\}^T[B]^T[B]\{\hat{d}\} d\hat{x} - \{\hat{d}\}^T\{\hat{P}\} - \int_0^L \{\hat{d}\}^T[N]^T\{T_y\} b d\hat{x}$$

Minimization of Potential Energy

$$\frac{d\pi_p}{d\{\hat{d}\}} = EI\int_0^L [B]^T[B]\{\hat{d}\} d\hat{x} - \{\hat{P}\} - \int_0^L [N]^T\{T_y\} b d\hat{x} = 0$$

Stiffness Matrix by Minimum Potential Energy Approach

$$\left(EI\int_0^L [B]^T[B]d\hat{x}\right)\{\hat{d}\} = \{\hat{P}\} + \int_0^L [N]^T\{T_y\}bd\hat{x} = \{\hat{f}\}$$

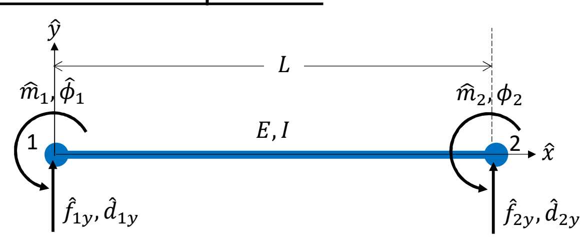
$[K]$ **Local Stiffness Matrix for Beam Element**

$$\begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2y} \\ \hat{m}_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} \hat{d}_{1y} \\ \hat{\phi}_1 \\ \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix}$$

Galerkin's Method to Derive Beam Element Equations in Local Coordinates

Step 4: Derive the Element Stiffness Matrix and Equations

- Direct Approach
- Minimum Potential Energy
- Galerkin's Method



Differential Equation for Beam

$$\left(EI\frac{d^4\hat{v}}{d\hat{x}^4}\right) + w(\hat{x}) = 0$$

E: Young's modulus
I: moment of inertia

Galerkin’s Method to Derive Beam Element Equations in Local Coordinates

$$\int_0^L \left(EI\frac{d^4\hat{v}}{d\hat{x}^4} + w(\hat{x})\right) N_i\, d\hat{x} = 0, \qquad i = 1, 2, 3, 4$$

$$\int_0^L EI\frac{d^4\hat{v}}{d\hat{x}^4} N_i\, d\hat{x} = \int_0^L EI \boxed{\frac{d^2\hat{v}}{d\hat{x}^2}}\overbrace{\boxed{\frac{d^2 N_i}{d\hat{x}^2}}}^{[B]} d\hat{x} + EI\left[\overbrace{\boxed{\frac{d^3\hat{v}}{d\hat{x}^3}}}^{V/EI} N_i - \overbrace{\boxed{\frac{d^2\hat{v}}{d\hat{x}^2}}}^{M/EI}\frac{dN_i}{d\hat{x}}\right]_0^L$$

$$\overbrace{\frac{d^2\hat{v}}{d\hat{x}^2}}^{[B]\{\hat{d}\}} = \underbrace{\left[\frac{12\hat{x} - 6L}{L^3} \quad \frac{6\hat{x}L - 4L^2}{L^3} \quad \frac{-12\hat{x} + 6L}{L^3} \quad \frac{6\hat{x}L - 4L^2}{L^3}\right]}_{[B]}\{\hat{d}\}$$

$$\left(\int_0^L [B]^T EI [B] d\hat{x}\right)\{\hat{d}\} + \int_0^L N_i w(\hat{x})\hat{x} + \left[\hat{V}N_i - \hat{M}\frac{dN_i}{d\hat{x}}\right]_0^L = 0$$

$$\left(\int_0^L [B]^T EI [B] d\hat{x}\right)\{\hat{d}\} = -\int_0^L N_i w(\hat{x})\hat{x} - \left[\hat{V}N_i - \hat{M}\frac{dN_i}{d\hat{x}}\right]_0^L$$

Galerkin’s Method to Derive Beam Element Equations in Local Coordinates

$$\left(\int_0^L [B]^T EI [B] d\hat{x}\right)\{\hat{d}\} = -\int_0^L N_i w(\hat{x})\hat{x} - \boxed{\left[\hat{V}N_i - \hat{M}\frac{dN_i}{d\hat{x}}\right]_0^L}$$

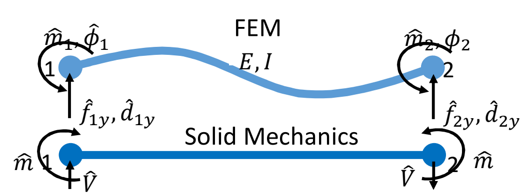
$[N]_0 = [1 \quad 0 \quad 0 \quad 0]$

$[N]_L = [0 \quad 0 \quad 1 \quad 0]$

$[dN/d\hat{x}]_0 = [0 \quad 1 \quad 0 \quad 0]$

$[dN/d\hat{x}]_L = [0 \quad 0 \quad 0 \quad 1]$

$$\boxed{\begin{Bmatrix}0\\0\\0\\1\end{Bmatrix}\hat{m}_2 - \begin{Bmatrix}0\\1\\0\\0\end{Bmatrix}\hat{m}_1 - \begin{Bmatrix}0\\0\\1\\0\end{Bmatrix}\hat{f}_{2y} + \begin{Bmatrix}1\\0\\0\\0\end{Bmatrix}\hat{f}_{1y}}$$



Galerkin's Method to Derive Beam Element Equations in Local Coordinates

$$\underbrace{\left(\int_0^L [B]^T EI[B]d\hat{x}\right)}_{[K]}\{\hat{d}\} = \boxed{-\int_0^L N_i w(\hat{x})d\hat{x}}_{\text{Surface Force}} + \begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2y} \\ \hat{m}_2 \end{Bmatrix}$$

$[K]$ **Local Stiffness Matrix for Beam Element**

$$\begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2y} \\ \hat{m}_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} \hat{d}_{1y} \\ \hat{\phi}_1 \\ \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix}$$

Summary

- Element Equations obtained by Minimum PE as well as Galerkin's Method.
- The details of each method are presented in the lecture.

Further Readings:
Sections 4-7 and 4-8 from "A first course in finite element" by Logan