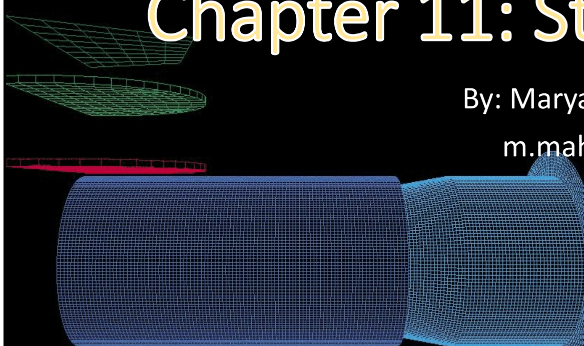
Chapter 11: Structural Dynamics

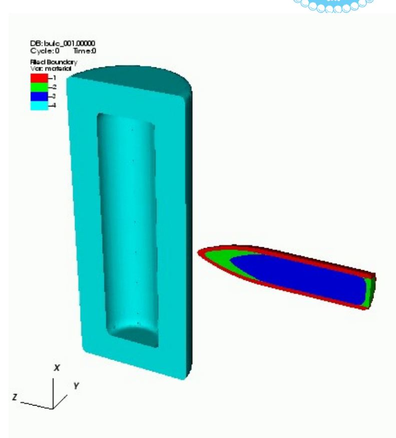
By: Maryam Mahnama, PhD
m.mahnama@ut.ac.ir



Motivation

- Time is the 4^{th} dimension.
- Each phenomenon can be studies in time transition.
- This dimension can be considered in finite element equations for each element.

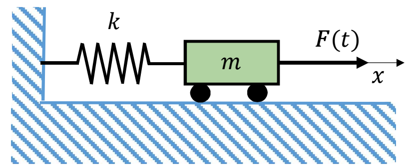
How time-dependent parameters are introduced into finite element simulation?



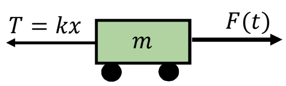
Basic Concepts

Single-Degree-of-Freedom mass-spring system

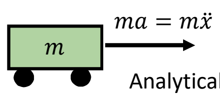
$F(t)$: time-dependent force



Free-body diagram:



Newton's second law:



$$F(t) - kx = m\ddot{x}$$

$$m\ddot{x} + kx = F(t)$$

2nd order differential equation

Free vibrations

Analytical Solution: Homogeneous + Particular parts

Approximate Numerical Solution in used FEM

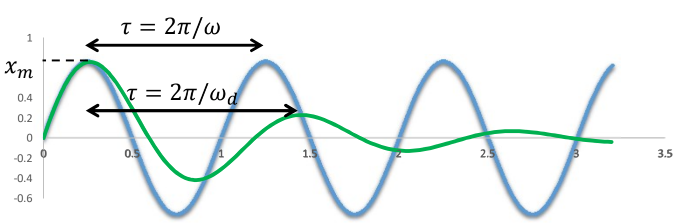
Basic Concepts

$$m\ddot{x} + kx = 0$$

$$\ddot{x} + \omega^2 x = 0$$

$$\omega = \sqrt{\frac{k}{m}}$$

Natural Circular Frequency

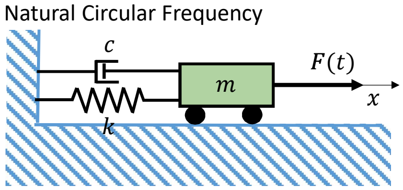


$x(t) = \boxed{x_m} \sin \omega t$

Vibration Amplitude

Does not have a unique answer

$x(t) = x_m e^{-\gamma} \sin \omega_d t$



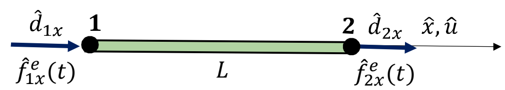
$$m\ddot{x} + c\dot{x} + kx = F(t)$$

$$m\ddot{x} + c\dot{x} + kx = 0$$

Derivation of Bar Element Formulation

Time-dependent (dynamic) stress analysis of Bar element

Step 1: Select Element Type



Element length: L

Cross-sectional area: A

Density: ρ

Step 2: Select Displacement Function

$$\boxed{\hat{u} = a_0 + a_1 \hat{x}} \Rightarrow \hat{u} = \underbrace{\left(1 - \frac{\hat{x}}{L}\right)}_{N_1(\hat{x})} \hat{d}_{1x} + \underbrace{\frac{\hat{x}}{L}}_{N_2(\hat{x})} \hat{d}_{2x}$$

Derivation of Bar Element Formulation

Step 3: Define the Strain/Disp. and Stress/Strain Relationships

The strain/displacement relationship

$$\epsilon_x = \frac{d\hat{u}}{d\hat{x}} = \frac{d[N]}{d\hat{x}}\{d\}$$

$$\epsilon_x = \left(\frac{\hat{d}_{2x} - \hat{d}_{1x}}{L}\right)$$

$$= \underbrace{\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}}_{[B]} \underbrace{\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}}_{\{d\}}$$

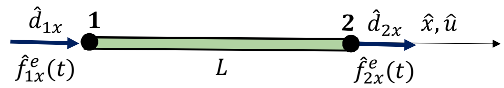
stress/strain relationship

$$\sigma_x = E\epsilon_x$$

$$\sigma_x = E[B]\{d\}$$

Derivation of Bar Element Formulation

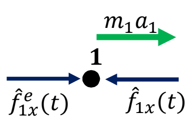
Step 4: Derive the Element Stiffness and Mass Matrices and Equations

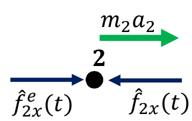


The forces are not in equilibrium

Total mass = $\rho LA \Rightarrow m_1 = m_2 = \rho LA/2$

Newton's 2nd law:





$$\hat{f}_{1x}^{e}(t) - \hat{f}_{2x}(t) = m_1 \frac{\partial^2 \hat{d}_{1x}}{\partial t^2}$$

$$\hat{f}_{2x}^{e}(t) - \hat{f}_{2x}(t) = m_2 \frac{\partial^2 \hat{d}_{2x}}{\partial t^2}$$

$$\begin{Bmatrix} \hat{f}_{1x}^{e} \\ \hat{f}_{2x}^{e} \end{Bmatrix} = \begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} + \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial^2 \hat{d}_{1x}}{\partial t^2} \\ \dfrac{\partial^2 \hat{d}_{2x}}{\partial t^2} \end{Bmatrix}$$

Derivation of Bar Element Formulation

$$\begin{Bmatrix} \hat{f}_{1x}^{e} \\ \hat{f}_{2x}^{e} \end{Bmatrix} = \begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} + \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial^2 \hat{d}_{1x}}{\partial t^2} \\ \dfrac{\partial^2 \hat{d}_{2x}}{\partial t^2} \end{Bmatrix}$$

Lumped Mass Matrix:

$$\{\hat{m}\} = \frac{\rho AL}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

has diagonal terms only…

Easier computation

Less accuracy

$$\{\hat{f}^{e}\} = \{\hat{f}\} + [\hat{m}]\{\ddot{\hat{d}}\}$$

$$\{\hat{f}^{e}\} = [\hat{k}]\{\hat{d}\} + [\hat{m}]\{\ddot{\hat{d}}\}$$

Bar element stiffness matrix in local coordinates

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \underbrace{\frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{[\hat{k}]} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

Derivation of Bar Element Formulation

$$\begin{Bmatrix} \hat{f}^e_{1x} \\ \hat{f}^e_{2x} \end{Bmatrix} = \begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} + \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial^2 \hat{d}_{1x}}{\partial t^2} \\ \dfrac{\partial^2 \hat{d}_{2x}}{\partial t^2} \end{Bmatrix}$$

$$\{\hat{f}^e\} = \{\hat{f}\} + [\widehat{m}]\{\ddot{\hat{d}}\}$$

$$\{\hat{f}^e\} = [\hat{k}]\{\hat{d}\} + [\widehat{m}]\{\ddot{\hat{d}}\}$$

Consistent Mass Matrix

- Principle of virtual work
  Minimum Potential Approach
- D' Alembert's Principle

Mass is considered as a body force acting on the volume of the body:

$$\{X^e\} = -\rho\{\ddot{u}\}$$

$$\{f_b\} = \iiint_V [N]^T\{X^e\}\,dV$$

Bar element stiffness matrix in local coordinates

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \underbrace{\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{[\hat{k}]} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

$$\{\hat{u}\} = [N]\{\hat{d}\}$$

$$\} = [N]\{\dot{\hat{d}}\}$$

$$\} = [N]\{\ddot{\hat{d}}\}$$

$$\{f_b\} = -\underbrace{\iiint_V \rho[N]^T[N]dV}_{[\widehat{m}]}\{\ddot{\hat{d}}\}$$

Consistent Mass Matrix $[\widehat{m}]$

Derivation of Bar Element Formulation

- <u>Consistent Mass Matrix for Bar Element</u>

$$[\widehat{m}] = -\iiint_V \rho \begin{Bmatrix} 1 - \dfrac{\hat{x}}{L} \\ \dfrac{\hat{x}}{L} \end{Bmatrix} \begin{bmatrix} 1 - \dfrac{\hat{x}}{L} & \dfrac{\hat{x}}{L} \end{bmatrix} dV \;\Rightarrow\; [\widehat{m}] = -\rho A \int_0^L \begin{Bmatrix} 1 - \dfrac{\hat{x}}{L} \\ \dfrac{\hat{x}}{L} \end{Bmatrix} \begin{bmatrix} 1 - \dfrac{\hat{x}}{L} & \dfrac{\hat{x}}{L} \end{bmatrix} d\hat{x}$$

$$[\widehat{m}] = -\rho A \int_0^L \begin{bmatrix} \left(1 - \dfrac{\hat{x}}{L}\right)^2 & \dfrac{\hat{x}}{L}\left(1 - \dfrac{\hat{x}}{L}\right) \\ \dfrac{\hat{x}}{L}\left(1 - \dfrac{\hat{x}}{L}\right) & \left(\dfrac{\hat{x}}{L}\right)^2 \end{bmatrix} d\hat{x} \;\Rightarrow\; [\widehat{m}] = -\frac{\rho A L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Derivation of Bar Element Formulation

- **Step 5: Assemble the Element Equations and Introduce Boundary Conditions**

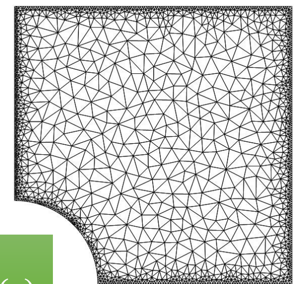
Interelement continuity of displacement

Interelement continuity of acceleration

$\{F(t)\} = [K]\{d\} + [M]\{\ddot{d}\}$

Space discretization

Time discretization



$$\{F\} = \sum_{e=1}^{N} \{f^{(e)}\}$$

Global Force Vector

$$[K] = \sum_{e=1}^{N} [k^{(e)}]$$

Global Stiffness Matrix

$$[M] = \sum_{e=1}^{N} [m^{(e)}]$$

Global Mass Matrix

Summary

- The effect of mass or density should be considered in dynamic analysis.
- There are two main approaches for mass matrix calculation: Lumped mass matrix and consistent.
- Interelement continuity of both displacement and acceleration happens in dynamic analysis.
- Both space and time discretization should be applied in dynamic analysis.