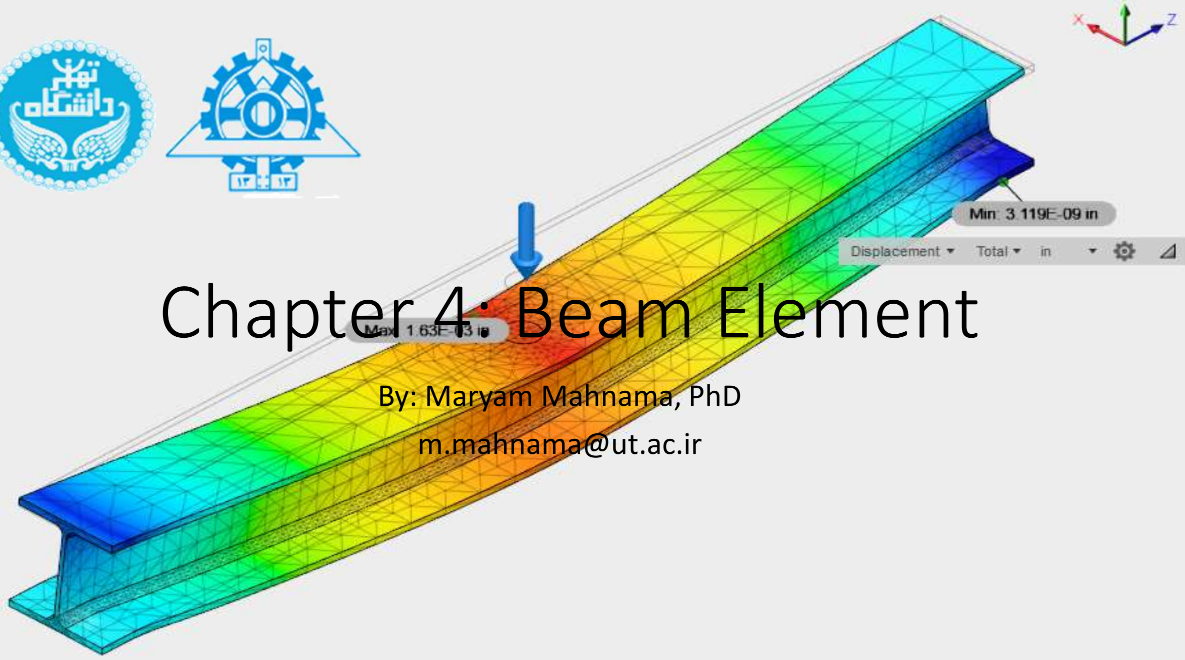

Chapter 4: Beam Element

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Contents of this lecture

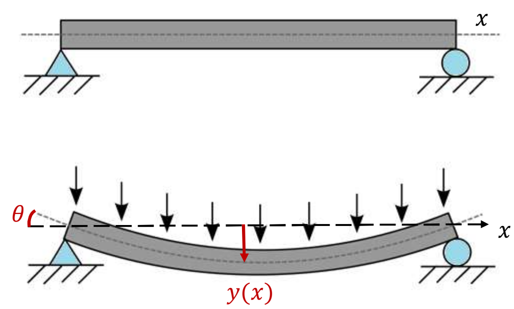
- Definitions of Beam element
- Statement of D.E. for Beam Element
- Displacement Function
- Strain/Displacement relations
- Derivation of Local Stiffness Matrix

Definition

- A **beam** is

Euler-Bernouli beam

a long, slender structural member generally subjected to transverse loading that produces significant bending effects as opposed to twisting or axial effects.

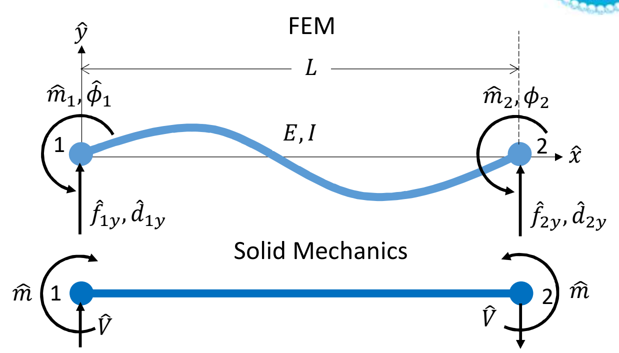


- Measure of bending deformation: transverse displacement + rotation.

DOFs to describe this component are **transverse displacement** and **rotation**

Beam Element Specifications

- Local coordinate system: $(\hat{x}, \hat{y})$
- Number of nodes: 2
- DOFs per node i:
  - Transverse displacement $\hat{d}_{iy}$
  - Rotation $\hat{\phi}_i$
- Local forces per node
  - Local nodal force $\hat{f}_{iy}$
  - Local bending moment $\hat{m}_i$



Sign Conventions:
1. Moments are positive in the counterclockwise direction.
2. Rotations are positive in the counterclockwise direction.
3. Forces are positive in the positive $\hat{y}$ direction.
4. Displacements are positive in the positive $\hat{y}$ direction

Differential Equation for Beam Element

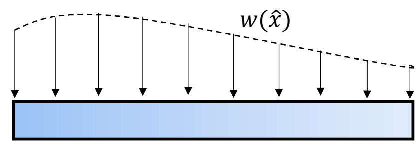
- Euler-Bernouli Beam Theory:

Plane cross sections perpendicular to the longitudinal centroidal axis of the beam before bending occurs remaining plane and perpendicular to the longitudinal axis after bending occurs.

Differential Equation for Beam

$$\frac{d^2}{d\hat{x}^2}\left(EI\frac{d^2\hat{v}}{d\hat{x}^2}\right) = -w(\hat{x})$$

E: Young's modulus
I: moment of inertia



Can be employed in weighted residual methods, such as Galerkin's method.

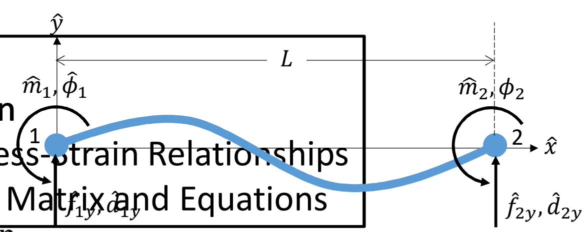
For Constant EI

$$EI\frac{d^4\hat{v}}{d\hat{x}^4} = -w(\hat{x})$$

Stiffness matrix for Beam element in local coordinate

Step 1: Define Element Type
Step 2: Select Displacement Function
Step 3: Strain-Displacement and Stress-strain Relationships
Step 4: Derive the Element Stiffness Matrix and Equations

# of DOF in element: 2 displacements
Along $\hat{y}$ and 2 rotations ⇒ 4 DOFs
⇒ 4 parameters in displacement function



$$\hat{v}(\hat{x}) = a_0 + a_1\hat{x} + a_2\hat{x}^2 + a_3\hat{x}^3$$

Assumption: small rotation ⟹ $\hat{\phi} = \frac{d\hat{v}}{d\hat{x}}$

B.C.s

$$\hat{v}(0) = \hat{d}_{1y} = a_0$$
$$\hat{v}(L) = \hat{d}_{2y} = a_0 + a_1L + a_2L^2 + a_3L^3$$
$$\frac{d\hat{v}}{d\hat{x}}(0) = \hat{\phi}_1 = a_1$$
$$\frac{d\hat{v}}{d\hat{x}}(L) = \hat{\phi}_2 = a_1 + 2a_2L + 3a_3L^2$$

Stiffness matrix for Beam element in local coordinate

$$\hat{v} = \overbrace{\boxed{\hat{d}_{1y}}}^{a_0} + \overbrace{\boxed{\hat{\phi}_1}}^{a_1}\hat{x} + \overbrace{\boxed{\left[-\frac{3}{L^2}\left(\hat{d}_{1y} - \hat{d}_{2y}\right) - \frac{1}{L}\left(2\hat{\phi}_1 + \hat{\phi}_2\right)\right]}}^{a_2}\hat{x}^2 + \underbrace{\boxed{\left[+\frac{2}{L^3}\left(\hat{d}_{1y} - \hat{d}_{2y}\right) + \frac{1}{L^2}\left(\hat{\phi}_1 + \hat{\phi}_2\right)\right]}}_{a_3}\hat{x}^3)$$

$$\{\hat{d}\} = \begin{Bmatrix} \hat{d}_{1y} \\ \hat{\phi}_1 \\ \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix}$$

$$\hat{v} = \hat{d}_{1y}\overbrace{\boxed{\left(1 - 3\frac{\hat{x}^2}{L^2} + 2\frac{\hat{x}^3}{L^3}\right)}}^{N_1} + \hat{\phi}_1\overbrace{\boxed{\left(\hat{x} - 2\frac{\hat{x}^2}{L} + \frac{\hat{x}^3}{L^2}\right)}}^{N_2}$$

$$+\hat{d}_{2y}\overbrace{\boxed{\left(+3\frac{\hat{x}^2}{L^2} \; -2\frac{\hat{x}^3}{L^3}\right)}}^{N_3} + \hat{\phi}_2\overbrace{\boxed{\left(-\frac{\hat{x}^2}{L} + \frac{\hat{x}^3}{L^2}\right)}}^{N_4}$$

Shape Functions for Beam Element

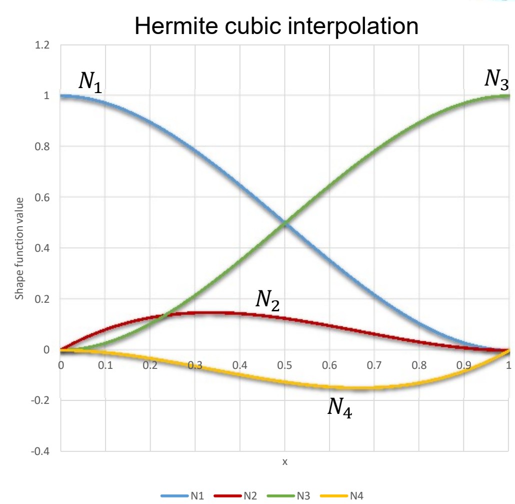
$$N_1 = \frac{1}{L^3}\left(L^3 - 3\hat{x}^2 L + 2\hat{x}^3\right)$$

$$N_2 = \frac{1}{L^3}\left(\hat{x}L^3 - 2\hat{x}^2 L^2 + \hat{x}^3 L\right)$$

$$N_3 = \frac{1}{L^3}\left(+3\hat{x}^2 L - 2\hat{x}^3\right)$$

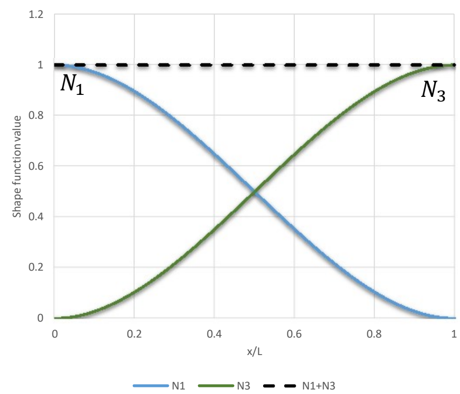
$$N_4 = \frac{1}{L^3}\left(-\hat{x}^2 L^2 + \hat{x}^3 L\right)$$

$N_1(0) = 1$
$N_1(L) = 0$
$dN_1/d\hat{x}(0) = 0$
$dN_1/d\hat{x}(L) = 0$
$N_2(0) = 0$
$N_2(L) = 0$
$dN_2/d\hat{x}(0) = 1$
$dN_2/d\hat{x}(L) = 0$
$N_3(0) = 0$
$N_3(L) = 1$
$dN_3/d\hat{x}(0) = 0$
$dN_3/d\hat{x}(L) = 0$
$N_4(0) = 0$
$N_4(L) = 0$
$dN_4/d\hat{x}(0) = 0$
$dN_4/d\hat{x}(L) = 1$



Shape Functions for Beam Element

$N_1 = \frac{1}{L^3}(L^3 - 3\hat{x}^2 L + 2\hat{x}^3) + N_3 = \frac{1}{L^3}(+3\hat{x}^2 L - 2\hat{x}^3) \Rightarrow 1$



Rigid-body displacement:

$$\hat{v}(\hat{x}) = N_1\hat{d} + N_2(0) + N_3\hat{d} + N_4(0) = \hat{d}$$
$$= (N_1 + N_3)\hat{d} = \hat{d} \rightarrow N_1 + N_3 = 1$$

Rigid-body rotation:

$$d\hat{v}/d\hat{x} = N_1'\hat{d} + N_2'\hat{\phi} + N_3'(\hat{d} + L\hat{\phi}) + N_4'\hat{\phi} = \hat{\phi}$$
$$= \underbrace{(N_1' + N_3')}_{0}\hat{d} + \underbrace{(N_2' + N_4' + LN_3')}_{1}\hat{\phi} = \hat{\phi}$$

Note: The shape functions for the Euler-Bernoulli beam have to be C[1]-continuous: Interelement continuity for displacement field (v) and its 1st derivative (dv/dx)

Stiffness matrix for Beam element in local coordinate

Step 3: Strain-Displacement and Stress-Strain Relationships

Step 1: Define Element Type

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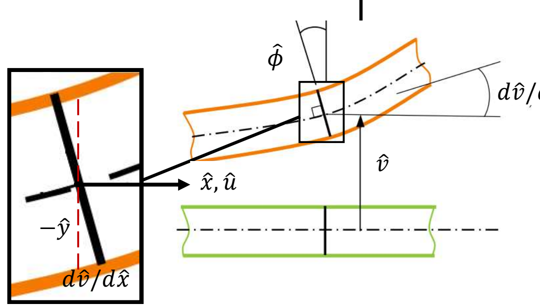
$$\epsilon_x(\hat{x}, \hat{y}) = \frac{d\hat{u}}{d\hat{x}}$$

$$\hat{u} = -\hat{y}\frac{d\hat{v}}{d\hat{x}}$$

$$\epsilon_x(\hat{x}, \hat{y}) = -\hat{y}\frac{d^2\hat{v}}{d\hat{x}^2}$$



The bending moment and shear force are related transverse displacement function

$$\hat{m}(\hat{x}) = EI\frac{d^2\hat{v}}{d\hat{x}^2} \qquad \hat{V}(\hat{x}) = EI\frac{d^3\hat{v}}{d\hat{x}^3}$$

Stiffness matrix for Beam element in local coordinate

Step 4: Derive the Element Stiffness Matrix and Equations

Step 1: Define Element Type

- Direct Equilibrium Approach:

Step 2: Select Displacement Function

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$$\hat{f}_{1y} = \hat{V} = EI\frac{d^3\hat{v}(0)}{d\hat{x}^3} = \frac{EI}{L^3}(12\hat{d}_{1y} + 6L\hat{\phi}_1 - 12\hat{d}_{2y} + 6L\hat{\phi}_2)$$

$$\boxed{\hat{m}_1 = -\hat{m}} = -EI\frac{d^2\hat{v}(0)}{d\hat{x}^2} = \frac{EI}{L^3}(6L\hat{d}_{1y} + 4L^2\hat{\phi}_1 - 6L\hat{d}_{2y} + 2L^2\hat{\phi}_2)$$

$$\boxed{\hat{f}_{2y} = -\hat{V}} = -EI\frac{d^3\hat{v}(L)}{d\hat{x}^3} = \frac{EI}{L^3}(-12\hat{d}_{1y} - 6L\hat{\phi}_1 + 12\hat{d}_{2y} - 6L\hat{\phi}_2)$$

$$\hat{m}_2 = \hat{m} = EI\frac{d^2\hat{v}(L)}{d\hat{x}^2} = \frac{EI}{L^3}(6L\hat{d}_{1y} + 2L^2\hat{\phi}_1 - 6L\hat{d}_{2y} + 4L^2\hat{\phi}_2)$$

Due to difference between FEM and Solid mechanics conversions

Stiffness matrix for Beam element in local coordinate

- Writing the equations in matrix notation:

Local Stiffness Matrix for Beam Element

$$\begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2y} \\ \hat{m}_2 \end{Bmatrix} = \boxed{\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}} \begin{Bmatrix} \hat{d}_{1y} \\ \hat{\phi}_1 \\ \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix}$$

Summary

- Beam element only sustains transverse loading and bending moment.
- Transverse displacement and rotation are the DOFs available for this element.
- Displacement function for 2-node beam element is a polynomial of 3rd order.
- Four cubic Hermitian shape functions can be extracted for this element.
- Employing Direct Stiffness Method, the stiffness matrix can be developed.

Further Readings:
Sections 4-1 from "A first course in finite element" by Logan