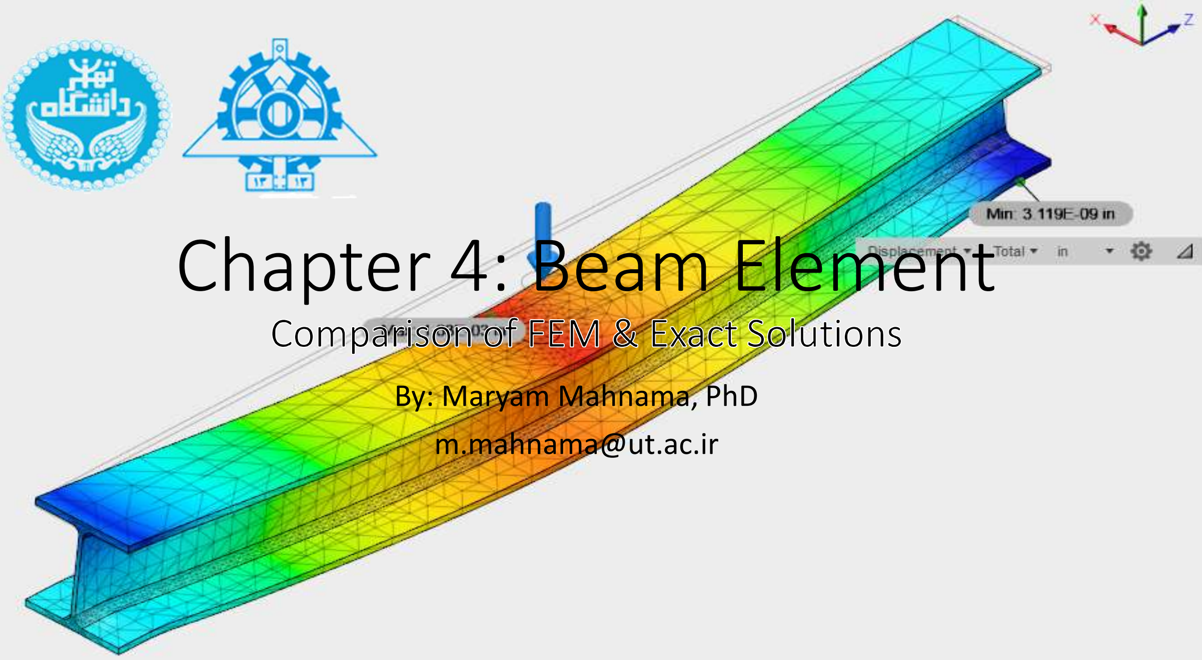

Chapter 4: Beam Element

Comparison of FEM & Exact Solutions

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Contents of this lecture

- Solution of an Example problem with Beam in Solid Mechanics
- Solution of an Example problem with Beam in FEM
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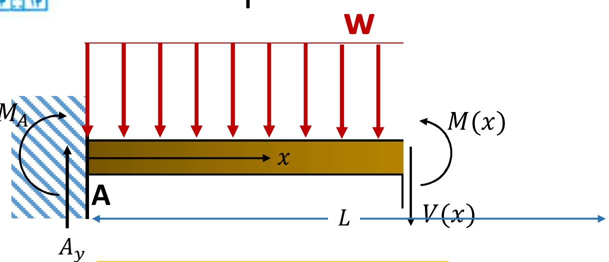
Example Problem in Solid Mechanics



In Solid Mechanics:

Equilibrium yields: $A_y = wL$

$M_A = -\frac{wL^2}{2}$

Moment-Curvature Eq: $y'' = \frac{M(x)}{EI}$

$$M(x) = \frac{wL^2}{2} - \frac{wx^2}{2} + wLx$$

$$EIy'' = -\frac{wL^2}{2} - \frac{wx^2}{2} + wLx$$

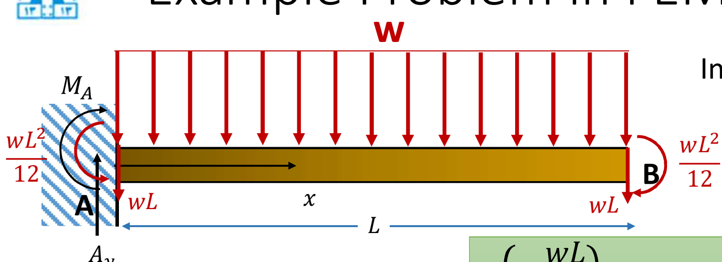
$$y(L) = \frac{-w\,L^4}{8EI}$$

$$y'(L) = \frac{-w\,L^3}{6EI}$$

$$y = \frac{1}{EI}\left(-\frac{wL^2x^2}{4} - \frac{wx^4}{24} + \frac{wLx^3}{6}\right)$$

$$EIy = -\frac{\;}{4} - \frac{\;}{24} + \frac{\;}{6} + \cancel{C_1 x} + \cancel{C_2}$$

Example Problem in FEM



In FEM:

$$\{\hat{f}\} = \begin{Bmatrix} -\frac{wL}{2} + A_y \\ \frac{wL^2}{12} + M_A \\ -\frac{wL}{2} \\ \frac{wL^2}{12} \end{Bmatrix}$$

$$\begin{Bmatrix} -\frac{wL}{2} \\ \frac{wL^2}{12} \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix}$$

$$\begin{Bmatrix} \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix} = \frac{L}{12EI}\begin{bmatrix} 4L^2 & 6L \\ 6L & 12 \end{bmatrix}\begin{Bmatrix} -\frac{wL}{2} \\ \frac{wL^2}{12} \end{Bmatrix}$$

$$\hat{d}_{2y} = \frac{-w\,L^4}{8EI}$$

$$\hat{\phi}_2 = \frac{-w\,L^3}{6EI}$$

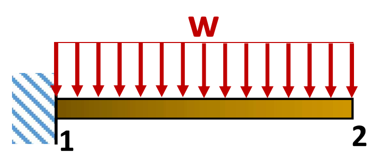
Comparison of Solid Mechanics & FEM

- $\hat{v}(x) = \frac{1}{L^3}(-2x^3 + 3x^2L)\hat{d}_{2y} + \frac{1}{L^3}(x^3L - x^2L^2)\hat{\phi}_2$
- $\hat{v}(x) = \frac{1}{L^3}(-2x^3 + 3x^2L)\left(-\frac{wL^4}{8EI}\right) + \frac{1}{L^3}(x^3L - x^2L^2)\left(-\frac{wL^3}{6EI}\right)$

FEM Solution: $\boxed{\hat{v}(x) = \frac{-wL}{24EI}(-2x^3 + 5x^2L)} \Rightarrow \hat{v}\left(\frac{L}{2}\right) = -\frac{16wL^4}{384EI}$

Exact Solution: $\boxed{y(x) = \frac{-w}{24EI}(x^4 - 4Lx^3 + 6L^2x^2)} \Rightarrow y\left(\frac{L}{2}\right) = -\frac{17wL^4}{384EI}$

Comparison of Solid Mechanics & FEM

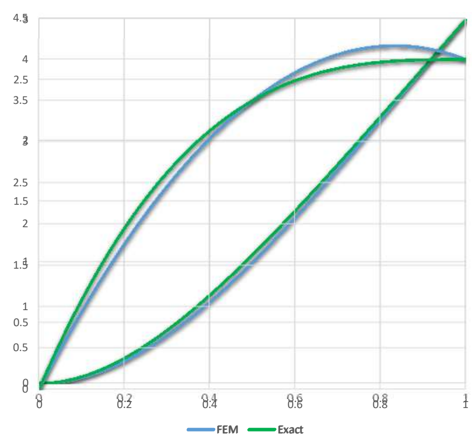


$$\boxed{\hat{v}(x) = \frac{-wL}{24EI}(-2x^3 + 5x^2L)}$$

$$\boxed{y(x) = \frac{-w}{24EI}(x^4 - 4Lx^3 + 6L^2x^2)}$$

$$\boxed{\frac{d\hat{v}}{dx}(x) = \frac{-wL}{24EI}(-6x^2 + 10xL)}$$

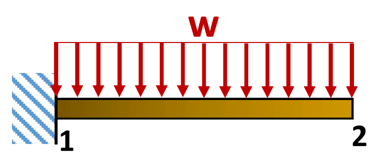
$$\boxed{y'(x) = \frac{-w}{24EI}(4x^3 - 12Lx^2 + 12L^2x)}$$



Comparison of Solid Mechanics & FEM

- The beam theory predicts a 4th polynomial expression for y for a beam subjected to uniformly distributed loading, while the FE solution $\hat{v}$ assumes a 3rd order displacement behavior in each beam element under all load conditions.
- Displacements $\hat{v}$ are lower than by the beam theory except at the nodes.
- FEM forces the beam into specific modes of displacement and effectively yields a stiffer model than the actual structure.
- FEM predicts a stiffer structure than the actual one.
- Using more elements, FEM solution converges to beam theory solution.

Comparison of Solid Mechanics & FEM

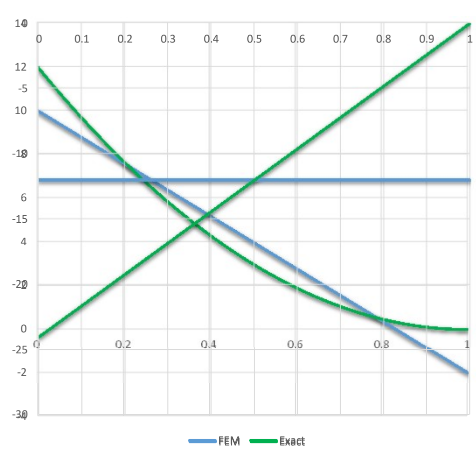


$$\frac{M(x)}{EI} = \frac{d^2\hat{v}}{dx^2}(x) = \frac{-wL}{24EI}(-12x + 10L)$$

$$\frac{M(x)}{EI} = y''(x) = \frac{-w}{24EI}(12x^2 - 12Lx + 12L^2)$$

$$\frac{V(x)}{EI} = \frac{d^3\hat{v}}{dx^3}(x) = \frac{wL}{24EI}(12)$$

$$\frac{V(x)}{EI} = y'''(x) = \frac{-w}{24EI}(24x - 12L)$$



Comparison of Solid Mechanics & FEM

- FE solution does not predict the bending moment as well as it does the displacement.
- The best approximation for bending moment appears at the midpoint of the element.
- The best approximation for shear force is at the midpoint of the element.

How can we improve FE solution

1. Using more elements in the model (refine the mesh)
2. Using a higher-order element, such as a fifth-order approximation for the displacement function,

$$\boxed{\hat{v}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5}$$

We need an element with three nodes
(with an extra node at the middle of the element).