Chapter 10: Thermal Stress

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Motivation

- We know that different types of loading, result in internal forces in the body.
- We have also explored the effects of body forces and surface forces.
- Can other types of external energies, such as thermal, electrical or magnetic energies, be transformed into some forces in the body?
- If so, how can we calculate such forces?
- Here, in this lecture, the effects of thermal energies on internal forces (stresses) are to be calculated.

Formulation of the Thermal Stress

A free beam under a temperature change of ΔT, undergoes an axial elongation:

$$\delta_T = \boxed{\alpha} L \Delta T$$

Coefficient of Thermal Expansion

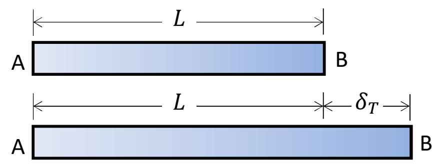
So, the axial strain is

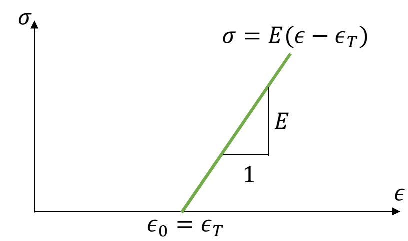
$$\epsilon_T = \frac{\delta_T}{L} = \alpha \Delta T$$

but the stress in the beam is zero.

$$\sigma_T = 0$$

For statically-indetermined structures, a temperature change will result to stress in body:





Formulation of the Thermal Stress

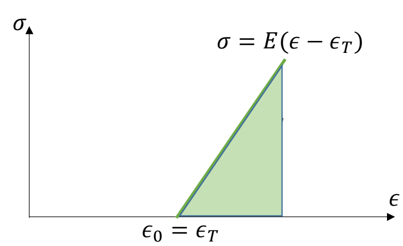
For the one-dimensional problem, we have,

$$\epsilon_x = \frac{\sigma_x}{E} + \epsilon_T$$

In general

$$\{\epsilon\} = [D]^{-1}\{\sigma\} + \{\epsilon_T\}$$

$$\{\sigma\} = [D](\{\epsilon\} - \{\epsilon_T\})$$



The strain energy per unit volume (called strain energy density) is the area under the $(\sigma - \epsilon)$ diagram:

$$u_0 = \frac{1}{2}\{\sigma\}^T(\{\epsilon\} - \{\epsilon_T\})$$

$$u_0 = \frac{1}{2}(\{\epsilon\}^T - \{\epsilon_T\}^T)[D](\{\epsilon\} - \{\epsilon_T\})$$

Formulation of the Thermal Stress

- Strain energy will be

$$U = \int u_0 dV = \int \frac{1}{2}(\{\epsilon\}^T - \{\epsilon_T\}^T)[D](\{\epsilon\} - \{\epsilon_T\})\, dV$$

$$= \int \frac{1}{2}(\{d\}^T[B]^T - \{\epsilon_T\}^T)[D]([B]\{d\} - \{\epsilon_T\})\, dV$$

$$U = \int \Big(\underbrace{\boxed{\frac{1}{2}(\{d\}^T[B]^T[D][B]\{d\}}}_{\text{usual strain energy due to stress produced from mechanical loading}} - \underbrace{\boxed{\{d\}^T[B]^T[D]\{\epsilon_T\}} - \boxed{\{\epsilon_T\}^T[D][B]\{d\}}}_{\text{Identical terms}} + \underbrace{\boxed{\{\epsilon_T\}^T[D]\{\epsilon_T\})}}_{\text{Constant}} dV$$

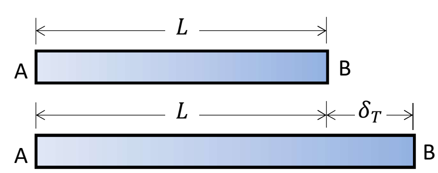
- Neglecting body force, surface force and nodal forces, potential energy is U, then:

$$\frac{\partial \Pi}{\partial \{d\}} = \underbrace{\boxed{\int [B]^T[D][B]dV}}_{\{k\}\text{: stiffness matrix}} \{d\} - \underbrace{\boxed{\int [B]^T[D]\{\epsilon_T\}dV}}_{\{f_T\}\text{: force vector due to temperature change in the element}} = 0$$

Thermal Stress in Bar Element

- We get the formulation as: $\{f_T\} = \int [B]^T[D]\{\epsilon_T\}dV$

Assumptions: 1-D isotropic material $\{\epsilon_T\} = \{\epsilon_{xT}\} = \{\alpha\, \Delta T\}$



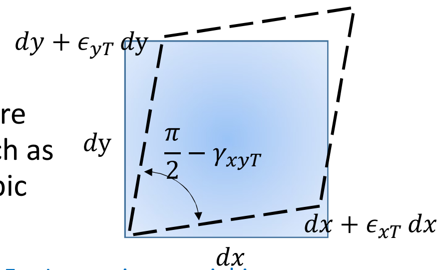
$$\Rightarrow \{f_T\} = \int [B]^T[D]\{\alpha\, \Delta T\}dV$$

$$\{f_T\} = \int_0^L \begin{Bmatrix} -\dfrac{1}{L} \\ \dfrac{1}{L} \end{Bmatrix} E\{\alpha\, \Delta T\}\, dx = \begin{Bmatrix} -E\alpha\Delta T \\ E\alpha\Delta T \end{Bmatrix}$$

Thermal Stress in 2D Elements

For the two-dimensional thermal stress problem,

- Two normal strains, ϵ_{xT} and ϵ_{yT} along with
- A shear strain γ_{xyT} due to the change in temperature because of the different mechanical properties (such as $E_x \neq E_y$) in the x and y directions for the anisotropic material



For Anisotropic material:

$$\{\epsilon_T\} = \begin{Bmatrix} \epsilon_{xT} \\ \epsilon_{yT} \\ \gamma_{xyT} \end{Bmatrix}$$

For Isotropic material in plane stress:

$$\{\epsilon_T\} = \begin{Bmatrix} \alpha\Delta T \\ \alpha\Delta T \\ 0 \end{Bmatrix}$$

For Isotropic material in plane strain:

$$\{\epsilon_T\} = (1+\nu)\begin{Bmatrix} \alpha\Delta T \\ \alpha\Delta T \\ 0 \end{Bmatrix}$$

Example

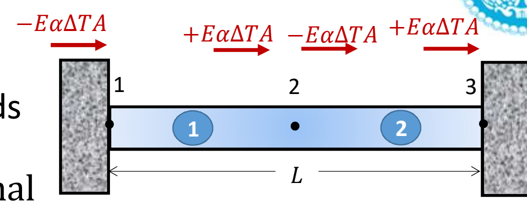
- In the case of a plane stress CST element, the thermal force vector is

$$\{f_T\} = \int [B]^T [D] \begin{Bmatrix} \alpha\Delta T \\ \alpha\Delta T \\ 0 \end{Bmatrix} dV$$

$$= \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}^T \frac{1}{1-\nu^2}\begin{bmatrix} \nu & 1 & 0 \\ 1 & \nu & 0 \\ 0 & 0 & \frac{1+\nu}{2} \end{bmatrix} tA = \frac{\alpha E \Delta T}{2(1-\nu)}\begin{Bmatrix} \beta_1 \\ \gamma_1 \\ \beta_2 \\ \gamma_2 \\ \beta_3 \\ \gamma_3 \end{Bmatrix}$$

This force vector should be added to body force, surface force and nodal force vectors.

Example



- For the bar element fixed at both ends under a temperature rise of 50°F, we want to find nodal forces and internal stress. $E = 30 \times 10^6 psi, \alpha = 7 \times \frac{10^{-6}}{°\text{F}}, A = 4\ in^2, L = 4\ ft$

equivalent nodal forces

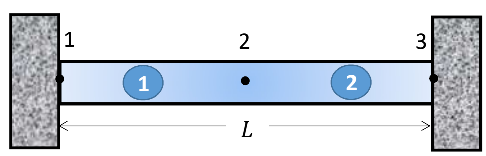
$$\underline{k}^{(1)} = \frac{AE}{L/2} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \frac{\text{lb}}{\text{in.}} \qquad \underline{f}^{(1)} = \begin{Bmatrix} -E\alpha TA \\ E\alpha TA \end{Bmatrix}$$

$$\{f_T\} = \begin{Bmatrix} -E\alpha\Delta TA \\ 0 \\ E\alpha\Delta TA \end{Bmatrix}$$

$$\underline{k}^{(2)} = \frac{AE}{L/2} \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \frac{\text{lb}}{\text{in.}} \qquad \underline{f}^{(2)} = \begin{Bmatrix} -E\alpha TA \\ E\alpha TA \end{Bmatrix}$$

Example

$$\{f_{concentrated}\} = \begin{Bmatrix} F_{1x} \\ 0 \\ F_{3x} \end{Bmatrix} \qquad \{f_T\} = \begin{Bmatrix} -E\alpha\Delta TA \\ 0 \\ E\alpha\Delta TA \end{Bmatrix}$$



$$\begin{Bmatrix} F_{1x} - E\alpha TA \\ 0 \\ F_{3x} + E\alpha TA \end{Bmatrix} = \frac{AE}{L/2} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{Bmatrix} \Rightarrow d_{2x} = 0$$

$$\Rightarrow F_{1x} = 42{,}000 \text{ lb} \qquad F_{2x} = 0 \qquad F_{3x} = -42{,}000 \text{ lb}$$

$$\sigma = \frac{42{,}000}{4} = 10{,}500 \text{ psi} \qquad \text{(compressive)}$$

Summary

- Plane Strain and Plane Stress Formulations can be used for LST Element.
- LST Element was defined and the displacement functions were extracted.
- Stiffness matrix for LST element can be derived using numerical integration.