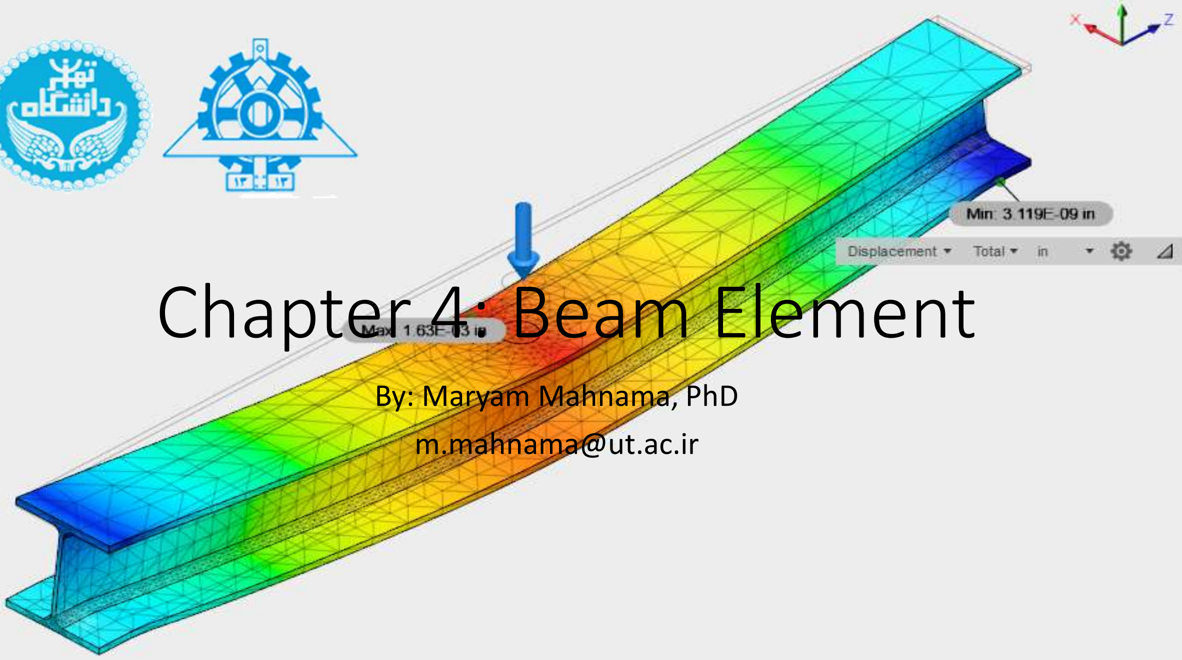

Chapter 4: Beam Element

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Motivation

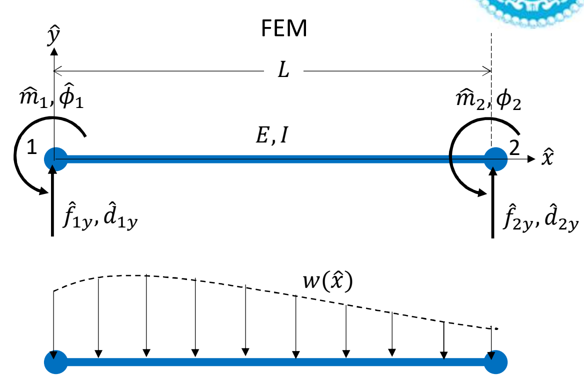
- Beam Element

$$\hat{v}(\hat{x}) = a_0 + a_1\hat{x} + a_2\hat{x}^2 + a_3\hat{x}^3$$

- Solid Mechanics

$$EI\frac{d^4\hat{v}}{d\hat{x}^4} = -w(\hat{x})$$

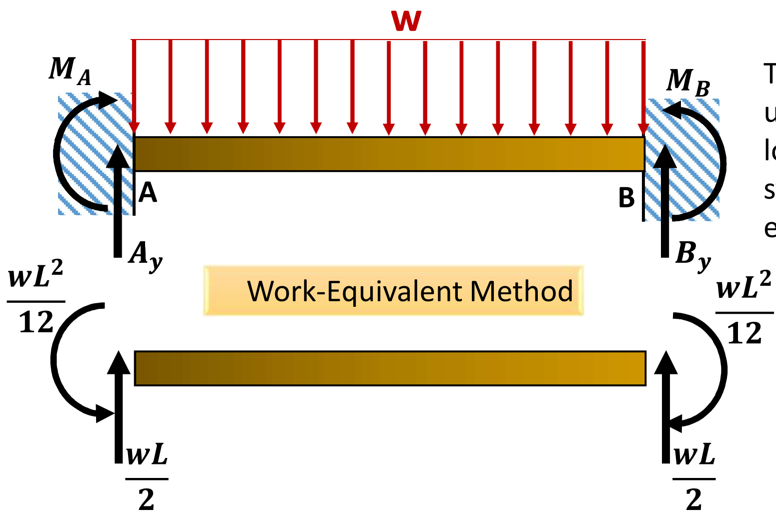
Is not a suitable answer for this differential equation



How can we get a reasonable result from FEM for this type of loading?

How can we replace the distributed load by nodal forces?

Motivation



The reactions on this beam under uniform distributed load can be obtained using symmetry + statics equilibrium relations:

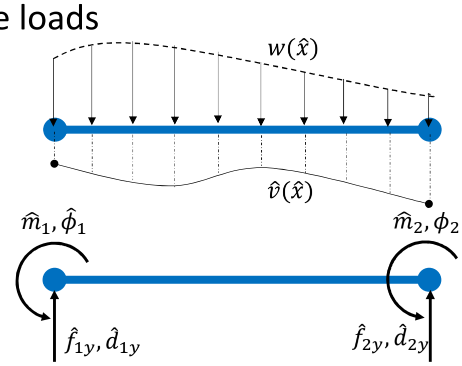
Work-Equivalent Method

- Replace a distributed load by a set of discrete loads

work of the distributed load $w(\hat{x})$ in going through the displacement field $\hat{v}(\hat{x})$

=

work done by nodal loads $\hat{f}_{iy}$ and $\hat{m}_i$ in going through nodal displacements $\hat{d}_{iy}$and $\hat{\phi}_i$ for arbitrary nodal displacements.

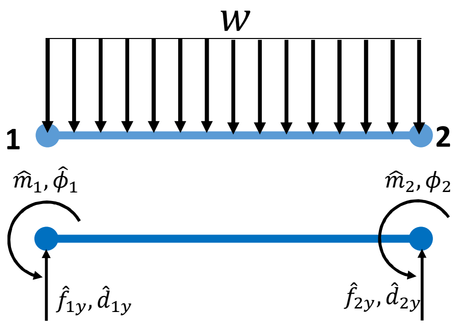


$$W_{distributed} = \int_0^L w(\hat{x})\hat{v}(\hat{x})d\hat{x}$$

$$W_{discrete} = \hat{f}_{1y}\,\hat{d}_{1y} + \hat{m}_1\,\hat{\phi}_1 + \hat{f}_{2y}\,\hat{d}_{2y} + \hat{m}_2\,\hat{\phi}_2$$

$$W_{distributed} = W_{discrete}$$

Example



$$\int_0^L w(\hat{x})\hat{v}(\hat{x})d\hat{x} = \hat{f}_{1y}\,\hat{d}_{1y} + \hat{m}_1\,\hat{\phi}_1 + \hat{f}_{2y}\,\hat{d}_{2y} + \hat{m}_2\,\hat{\phi}_2$$

$$\int_0^L w(\hat{x})\hat{v}(\hat{x})d\hat{x} = \int_0^L w\,[N]\{\hat{d}\}d\hat{x}$$

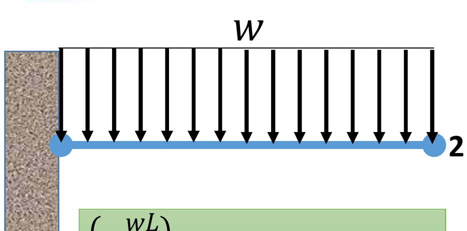
$$= -\frac{Lw}{2}(\hat{d}_{1y} - \hat{d}_{2y}) - \frac{L^2w}{2}(\hat{\phi}_1 + \hat{\phi}_2)$$

$$+ Lw(\hat{d}_{1y} - \hat{d}_{2y}) + \frac{L^2w}{3}(2\hat{\phi}_1 + \hat{\phi}_2) - \frac{L^2w}{2}(\hat{\phi}_1)$$

$$- Lw(\hat{d}_{1y})$$

$$= \underbrace{\boxed{-\frac{Lw}{2}}}_{\hat{f}_{1y}}\hat{d}_{1y} \underbrace{\boxed{-\frac{L^2w}{12}}}_{\hat{m}_1}\hat{\phi}_1 \underbrace{\boxed{-\frac{Lw}{2}}}_{\hat{f}_{2y}}\hat{d}_{2y} + \underbrace{\boxed{\frac{L^2w}{12}}}_{\hat{m}_2}\hat{\phi}_2$$

Example



Effect of Traction Force

Effect of Concentrated Force

$$\{F\} = \boxed{\begin{Bmatrix} -\dfrac{wL}{2} \\ -\dfrac{wL^2}{12} \\ -\dfrac{wL}{2} \\ \dfrac{wL^2}{12} \end{Bmatrix}} + \boxed{\begin{Bmatrix} F_1 \\ M_1 \\ 0 \\ 0 \end{Bmatrix}}$$

Boundary Conditions:
$\hat{d}_{1y} = 0$
$\hat{\phi}_1 = 0$

$$\begin{Bmatrix} -\dfrac{wL}{2} \\ \dfrac{wL^2}{12} \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix}$$

$$\begin{Bmatrix} -\dfrac{wL}{2} + F_1 \\ -\dfrac{wL^2}{12} + M_1 \\ -\dfrac{wL}{2} \\ \dfrac{wL^2}{12} \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} \hat{d}_{1y} \to 0 \\ \hat{\phi}_1 \to 0 \\ \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix}$$

Reduced Stiffness Matrix

Equivalency with MPE approach

- Recall from definition of traction force by minimum potential energy approach

$$\{\hat{f}_s\} \equiv \iint_{S_1} [N_s]^T \{\hat{X}_s\} dS$$

$$\underbrace{\boxed{\{\hat{f}_s\}^T \{\hat{d}\}}}_{W_{discrete}} \equiv \underbrace{\iint_{S_1} \underbrace{\boxed{[N_s]^T \{\hat{d}\}}}_{\hat{v}(\hat{x})} \underbrace{\boxed{\{\hat{X}_s\}}}_{w(\hat{x})} dS}_{W_{distributed}}$$

Summary

- In FEM we need to replace a distributed load by nodal forces.
- For this purpose we can use the equivalency of work done by distributed force to the work performed by replaced nodal forces.
- This approach yields the same result as the one obtained by minimum potential energy approach.